Divide by Zero and Conquer the World!

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Agenda

- A word of comfort
- How bad can bad get?
- Learn how to divide by zero
- Make computers safer and more accurate
- Summary

A Word of Comfort

- Dividing by zero is no more mysterious than finding the square root of a negative number
- Transreal arithmetic divides by zero using only accepted algorithms of arithmetic – you already know how to divide by zero!
- There is a machine proof that transreal arithmetic is consistent if real arithmetic is
- Transcomplex arithmetic has been developed
- Every real and complex result of mathematics stays the same, but there are some new non-finite results

Can Calculators Divide by Zero?

- If you have an electronic calculator then turn it on and stand up
- Pick a number and divide it by zero on your calculator
- If your calculator shows an error or has crashed then sit down
- If your calculator is still working then multiply the current answer by zero
- If your calculator shows an error or has crashed then sit down
- Is there anyone left standing?

Can Computers Divide by Zero?

- The computers and software I have designed can divide by zero
- Computers executing integer arithmetic cannot divide by zero
- Computers executing IEEE floating-point arithmetic cannot always divide by zero. They can produce infinities, but they also produce a class of objects that are all Not a Number (NaN)
- Replacing IEEE's minus zero with transreal nullity and replacing all of the NaNs with real numbers doubles the range of real numbers described by the floating-point bits, making the arithmetic more accurate

Can Computers Divide by Zero?

- The bridge of the missile cruiser, USS Yorktown, had networked computer control of navigation, engine monitoring, fuel control, machinery control, and damage control

Can Computers Divide by Zero?

- On September 21st, 1997, a sailor on the USS Yorktown entered a zero into a database field, causing a division by zero error which cascaded through the ship’s network, crashing every computer on the network, and leaving the ship dead in the water for 2 hours 45 minutes
- The world would be a safer place if computers, calculators and people could divide numbers by zero, getting a number as an answer
- Coincidentally, I worked out how to do this in 1997

Complex Numbers

People used to believe that it is impossible to find the square root of a negative number

- $\sqrt{-4} = ?$
- $2 \times 2 = 4$
- $(-2) \times (-2) = 4$

Complex Numbers

- Invent a new number $i = j = k = \sqrt{-1}$
- Use only accepted algorithms of arithmetic
- BUT change the way the algorithms are applied by making addition non-absorptive, i.e. keep real and imaginary sums separate

Complex Numbers

- For example, complex multiplication is defined by:

$$\begin{aligned}(a+ib)(c+id) &= a(c+id)+ib(c+id)\\ &= ac+iad+ibc+i^2bd\\ &= ac+iad+ibc+(-1)bd\\ &= (ac-bd)+i(ad+bc)\\ &= k_1+ik_2\end{aligned}$$

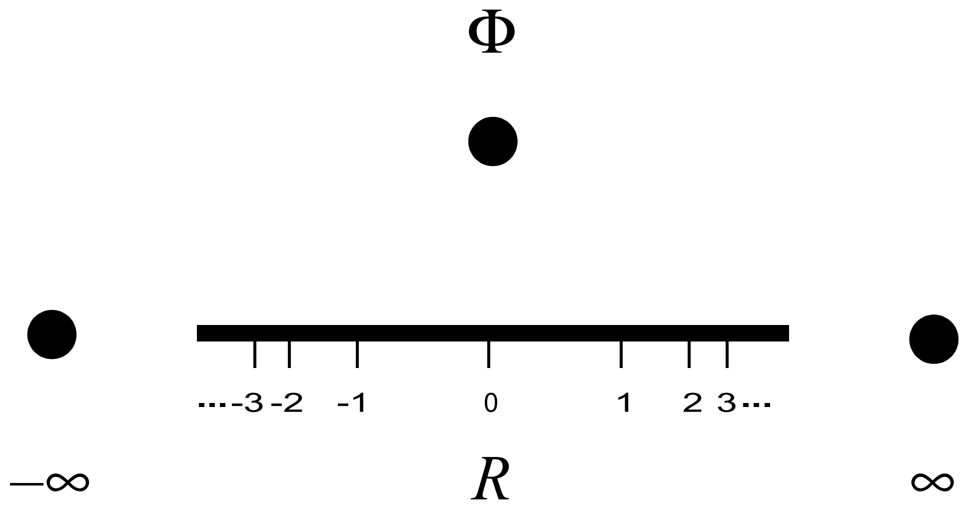
- Now $i2 \times i2 = i^2 4 = -1 \times 4 = -4$ so $\sqrt{-4} = i2$

Transreal Numbers

Invent some new numbers. For all $k > 0$ we define:

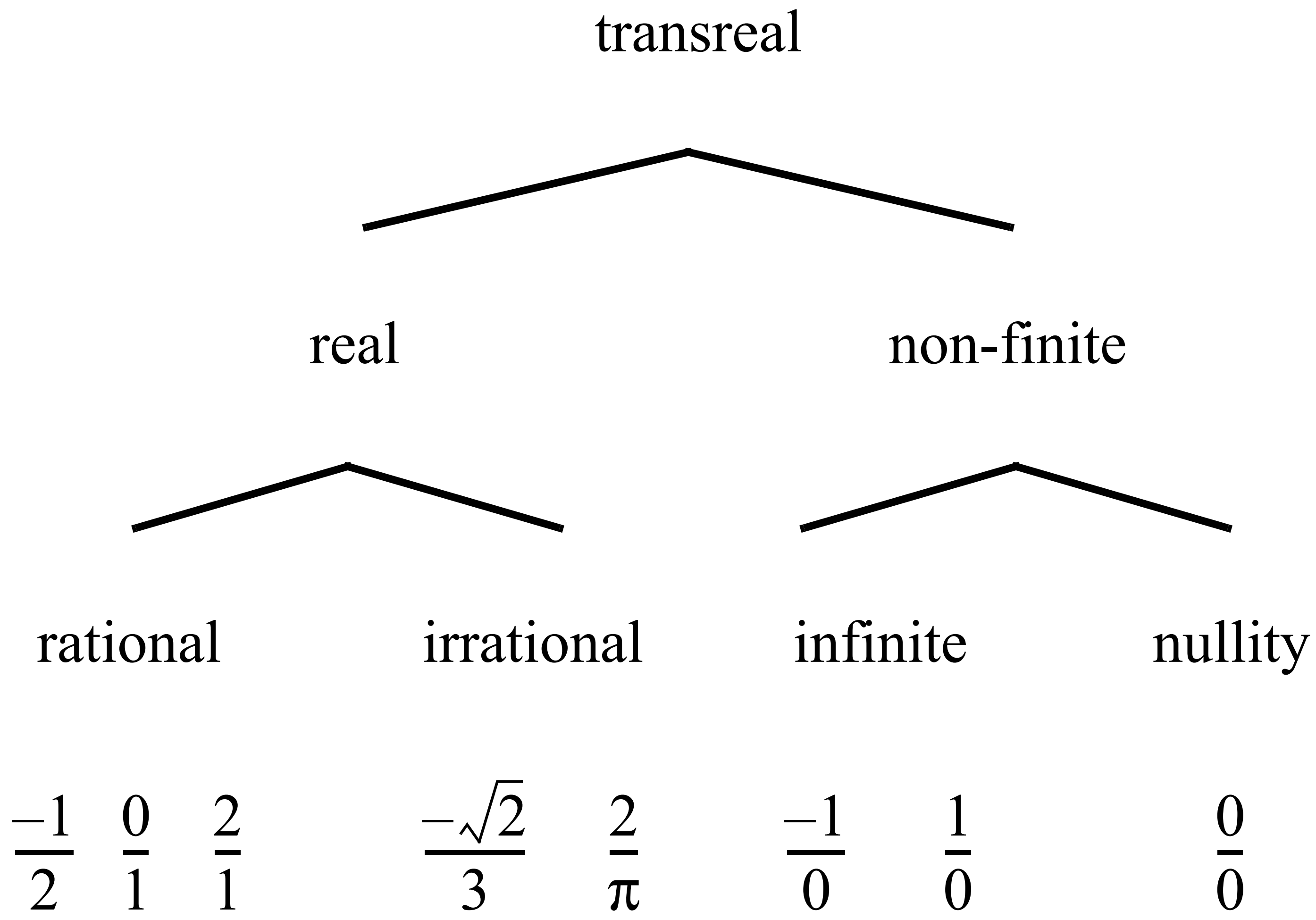
- $\infty = \frac{1}{0} \equiv \frac{k}{0}$
- $\Phi = \frac{0}{0}$
- $-\infty = \frac{-1}{0} \equiv \frac{-k}{0}$
- $0 = \frac{0}{1} \equiv \frac{0}{k} \equiv \frac{0}{-k}$

Transreal Numbers



- Positive infinity, ∞, is the biggest transreal number
- Negative infinity, −∞, is the smallest transreal number
- Nullity, Φ, is the only transreal number that is not negative, not zero, and not positive

Transreal Numbers



Transreal Fractions

A *transreal number* is a *transreal fraction* of the form $\frac{n}{d}$, where:

- n is the *numerator* of the fraction
- d is the *denominator* of the fraction
- n, d are *real numbers*
- $d \geq 0$
- Fractions with non-finite components simplify to the above form

Transreal Fractions

- An *improper transreal fraction*, $\frac{n}{-d}$, may have a negative denominator, $-d < 0$
- An improper transreal fraction is converted to a *proper transreal fraction* by multiplying both the numerator and denominator by minus one; or by negating both the numerator and the denominator, using subtraction; or it can be done, lexically, by moving the minus sign from the denominator to the numerator

$$\frac{n}{-d} = \frac{-1 \times n}{-1 \times (-d)} = \frac{-n}{-(-d)} = \frac{-n}{d}$$

Transreal Fractions

- Example: $\dfrac{2}{-3} = \dfrac{-1 \times 2}{-1 \times (-3)} = \dfrac{-2}{3}$
- Example: $\dfrac{0}{-1} = \dfrac{-0}{-(-1)} = \dfrac{0}{1}$
- Example: $\dfrac{x}{-y} = \begin{cases} \dfrac{x}{y} & : y = 0 \\ \dfrac{-x}{y} & : \text{otherwise} \end{cases}$

Transreal Multiplication

Two *proper transreal fractions* are multiplied like this:

- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
- Example: $3 \times \infty = \frac{3}{1} \times \frac{1}{0} = \frac{3 \times 1}{1 \times 0} = \frac{3}{0} = \infty$
- Example: $0 \times \infty = \frac{0}{1} \times \frac{1}{0} = \frac{0 \times 1}{1 \times 0} = \frac{0}{0} = \Phi$
- Example: $\frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$

Transreal Division

Two *proper transreal fractions* are divided like this:

- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$
- Example: $\infty \div 3 = \frac{1}{0} \div \frac{3}{1} = \frac{1}{0} \times \frac{1}{3} = \frac{1 \times 1}{0 \times 3} = \frac{1}{0} = \infty$
- Example:

$$\infty \div (-3) = \frac{1}{0} \div \frac{-3}{1} = \frac{1}{0} \times \frac{1}{-3} = \frac{1}{0} \times \frac{-1 \times 1}{-1 \times (-3)}$$

$$= \frac{1}{0} \times \frac{-1}{3} = \frac{1 \times (-1)}{0 \times 3} = \frac{-1}{0} = -\infty$$

Transreal Division

- Example: $\frac{1}{2} \div \frac{5}{3} = \frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$

Transreal Addition

Two *proper transreal fractions* are added like this:

- $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, except that:
- $(\pm\infty) + (\pm\infty) = \dfrac{\pm 1}{0} + \dfrac{\pm 1}{0} = \dfrac{(\pm 1) + (\pm 1)}{0}$ with the signs of the two terms $\pm\infty$ chosen independently, and with $+1$ corresponding to $+\infty$ and -1 corresponding to $-\infty$

Transreal Addition

- $(\pm\infty) + (\pm\infty) = \frac{\pm 1}{0} + \frac{\pm 1}{0} = \frac{(\pm 1) + (\pm 1)}{0}$

Examples:

- $\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1 + 1}{0} = \frac{2}{0} = \infty$
- $(-\infty) + (-\infty) = \frac{-1}{0} + \frac{-1}{0} = \frac{(-1) + (-1)}{0} = \frac{-2}{0} = -\infty$
- $\infty + (-\infty) = \frac{1}{0} + \frac{-1}{0} = \frac{1 + (-1)}{0} = \frac{0}{0} = \Phi$

Transreal Addition

- $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Examples:

- $\frac{2}{3} + \infty = \frac{2}{3} + \frac{1}{0} = \frac{2 \times 0 + 3 \times 1}{3 \times 0} = \frac{3}{0} = \infty$
- $\frac{2}{3} + \Phi = \frac{2}{3} + \frac{0}{0} = \frac{2 \times 0 + 3 \times 0}{3 \times 0} = \frac{0}{0} = \Phi$
- $\frac{2}{3} + \frac{4}{5} = \frac{2 \times 5 + 3 \times 4}{3 \times 5} = \frac{22}{15}$

Transreal Subtraction

Two *proper transreal fractions* are subtracted like this:

- $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$

Examples:

- $\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{0} + \frac{-1}{0} = \frac{1 + (-1)}{0} = \frac{1-1}{0} = \frac{0}{0} = \Phi$
- $\frac{1}{2} - \frac{3}{5} = \frac{1}{2} + \frac{-3}{5} = \frac{(1 \times 5) + (2 \times (-3))}{2 \times 5} = \frac{5 + (-6)}{10} = \frac{-1}{10}$

Transreal Arithmetic

- Transreal arithmetic is a superset of real arithmetic
- Transreal arithmetic is *total* – every operation of transreal arithmetic can be applied to any transreal numbers with the result being a transreal number
- Every syntactically correct sentence of transreal arithmetic is semantically correct so a program that compiles has no run time errors! This makes programs safer
- Real arithmetic is *partial* – it fails on division by zero and on each of the infinitely many mathematical consequences of division by zero and ordinary programs can have run time errors

Transreal Associativity

Transreal arithmetic is totally associative over addition and multiplication:

- $a + (b + c) = (a + b) + c$
- $a \times (b \times c) = (a \times b) \times c$

Transreal Commutativity

Transreal arithmetic is totally commutative over addition and multiplication:

- $a + b = b + a$
- $a \times b = b \times a$

Transreal Distributivity

Transreal arithmetic is only partially distributive:

$$a \times (b + c) = (a \times b) + (a \times c)$$

- If a is finite or nullity then a distributes over any $b + c$
- If a is infinity or minus infinity then a distributes if $b + c = \Phi$ or $b + c = 0$ or b and c have the same sign
- Two numbers have the same sign if they are both positive, both negative, both zero, or both nullity

Transreal Distributivity

Despite the fact that transreal arithmetic is only partially distributive, it is still a total arithmetic because we can always evaluate any arithmetical expression, including both of:

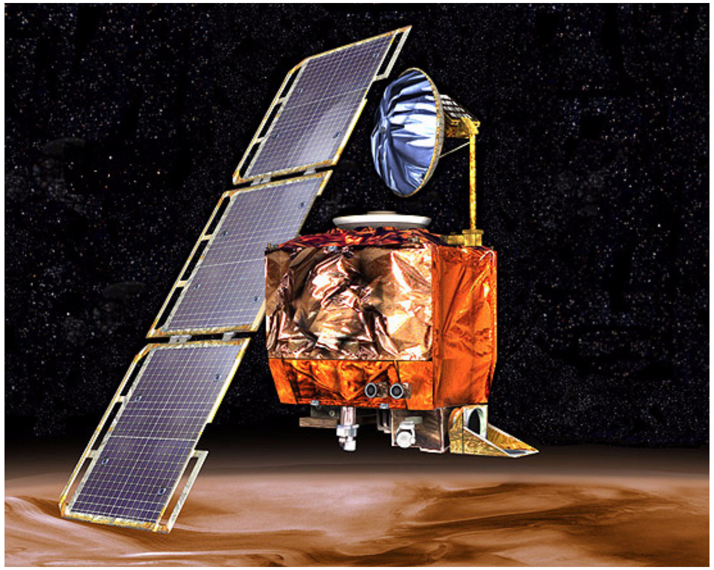
- $a \times (b + c)$
- $(a \times b) + (a \times c)$

It's just that these two expressions might, or might not, be equal!

- Computational paths generally bifurcate into a distributive and a non-distributive branch

Mars

- NASA’s Climate Orbiter, which cost $125 M, crashed into the surface of Mars on 23 September, 1999, because a computer program mixed up foot-pound-second units with metre-kilogram-second units



- “People sometimes make errors,” Edward Weiler, NASA's Associate Administrator for Space Science

Mars

- This bug could have been caught if the compiler had used dimensional analysis
- All ordinary type checking and ordinary dimensional analysis fail on division by zero – collapsing to a bottom state
- But all syntactically correct sentences of transarithmetic are semantically correct – which means that a compiler can always check, or generate code to check, every possible evaluation of the transarithmetic in any program
- How much would NASA pay for a compiler that can always apply dimensional analysis?

Moral

- The arithmetic you have just seen has been taught to 12 year old children in England
- These children understand infinity and nullity
- These children use an arithmetic that never fails
- What do you want for your children?
- What do you want for your computers?
- What do you want for your self?

Summary

- Transreal arithmetic uses only algorithms of real arithmetic so …
- You have known how to divide by zero since you were in secondary school. It is just that you and the whole of society had a mental block against dividing by zero
- Transreal floating-point hardware has no wasted states. This makes programs more accurate
- It is possible to design transreal computers so that any program that compiles has no run time errors. This makes programs safer