Naive Set-Theory Without Paradox!

Dr James Anderson FBCS CITP CSci

Transmathematics

- Trans - goes beyond the usual mathematics
- Aims to make all systems total, including:
- Aims to make all functions total
- Aims to make all operations closed

Transmathematics

- Trans-Boolean Logic
- Transalgebra
- Transreal analysis, including calculus
- Transcomplex analysis, no calculus yet
- Transcomputation

Naive-Set Theory

- The set $\{x \mid \phi(x)\}$ is defined by the class $\phi(x)$
- But classes are more general than sets so we are not surprised that naive set-theory is inconsistent
- The solution is to accept all of the notation and operations of naive set-theory as a class theory

Class Theory

- $\{x \mid \phi(x)\}$ is syntactic sugar for the class $\phi(x)$
- All other notations and operations of set theory are now generalised to classes

Interchangeability

- Interchangeability ($\doteq$) is an equivalence over all classes: $x \doteq y \Leftrightarrow x \in \{y\}$

Universes

- Universal Class (U): $U \doteq \{x \mid T\}$
- Universal Set (V): $V = \{x \mid x = x\}$
- Universal Antinomy (W): $W \doteq \{x \mid x \neq x\}$
- U is partitioned by V and W

Russell Class

- $R_U \doteq \{x_1 \mid x_2 \notin x_3\}$ with $x_1 \doteq x_2 \doteq x_3$

Russell Class

$$R_U \doteq \{x_1 \mid x_2 \notin x_3\}$$

- Suppose $R_U \doteq x_1$
- That is $R_U \in R_U$
- But $R_U \notin R_U$ because $x_2 \notin x_3$
- So $R_U \in R_U \Rightarrow R_U \notin R_U$

Russell Class

$$R_U \doteq \{x_1 \mid x_2 \notin x_3\}$$

- Suppose $R_U \doteq x_2 \doteq x_3$
- That is $R_U \notin R_U$
- But $R_U \in R_U$ because x_1
- So $R_U \notin R_U \Rightarrow R_U \in R_U$

Russell Paradox

- Combining implications we have a bi-implication that is the classical Russell Paradox:
- $R_U \in R_U \Leftrightarrow R_U \notin R_U$
- Russell assumed that the contradiction in the paradox proves that the set does not exist but it does exist, for us, as a class and we can work out its properties

Extensionality

- Axiom of Extensionality:

$$(x = y) \Rightarrow (z \in x \Rightarrow z \in y)$$

- Taking $x = y = z = R_U$
- Gives $(R_U = R_U) \Rightarrow (R_U \in R_U \Rightarrow R_U \in R_U)$
- But $R_U \in R_U \Leftrightarrow R_U \notin R_U$
- So $R_U \neq R_U$

Russell Antinomy

- $R_W \doteq R_U$

Russell Set

- $R_V = R_W \cap V$

Russell Truisms

- $R_W \in R_W$ is a gap
- $R_W \notin R_V$
- $R_V \in R_W$
- $R_V \notin R_V$

Atoms

- α_i is an atom iff it is a singleton set whose element is equal to α_i
- $\alpha_i = \{\alpha_i\}$
- Atoms are distinguished by extensionality
- One could define anti-atoms

Cardinality

- Assuming arbitrarily many atoms:
- $|U| \doteq |V| \doteq |W| \doteq |R_U| \doteq |R_V| \doteq |R_W|$
- There are enough sets to be cardinals of all classes
- $|V| = V$ is the greatest cardinal number

Powerclass

- $P(x) \doteq \{z \mid z \doteq x \setminus y\}$ for all $y \in U$
- x is given bottom up
- y is given top-down

Cantor's Demi-Proof

- Given the bijection: $f : A \to P(A)$
- For elements: $f(x) \doteq B_x$
- Cantor considers: $C \doteq \{x \mid (x \in A) \& (x \notin B_x)\}$
- Cantor proves: $|A| < |P(A)|$

Cantor's Demi-Proof

- Cantor ignored the branch of the proof with:
- $C' \doteq \{x \mid (x \in A) \& (x \in B_x)\}$
- Whence: $A \doteq P(A)$
- So: $|A| = |P(A)|$

Cantor's Demi-Proof

- Axiom of Singleton Classes: $x \Leftrightarrow \{x\}$

- When the axiom does not hold it can be the case that: $|A| < |P(A)|$

- When the axiom holds: $|A| = |P(A)|$

Cantor's Demi-Proof

- When: $x \in \{U, V, W, R_U, R_V, R_W\}$
- $x \doteq P(x)$
- $|x| = |P(x)|$
- So there is a greatest cardinal number

Cantor’s Demi-Proof

- The greatest cardinal is said to be inconsistent with Zermelo-Fraenkel (ZF) Set-Theory

- Is ZF overly conservative?

- Is ZF inconsistent?

Von Neumann Ordinals

- $n_0 = 0 = \{\}$
- $n_1 = 1 = \{0\}$
- $n_2 = 2 = \{0,1\}$
- $\ldots$

Von Neumann Ordinals

- $n_i < n_j \Leftrightarrow n_i \in n_j$

Transordinals

- Bottom-up definition:
- Every ordinal is transordinal

Transordinals

- Top-down definitions:
- Transordinal infinity is: $\infty = V \setminus \{V\}$
- Transordinal nullity is: $\Phi \doteq W \setminus \{W\}$

Transordinals

- Transordinal infinity is the greatest cardinal and the greatest transordinal
- Transordinal nullity is unordered
- Thus all of the transarithmetics are extended

Burali-Forti

- The Burali-Forti Paradox establishes that it is impossible to construct the set of all ordinal numbers, being their ordinal type, by bottom-up methods

- Transordinal ∞ contains the set of all ordinals as a proper subset and it is the ordinal type of the ordinals, constructed by top-down methods

Reasoning

- If mixed bottom-up and top-down reasoning establishes more theorems than bottom up reasoning alone then:

- Competent AI is *scruffy* not *neat*

- Competent languages are untyped

- The axiom $V = L$ is false

Conclusion

- Class theory is just naive set theory with antinomies - it is pedagogically simple!
- Class theory is consistent with: the greatest cardinal; the greatest ordinal; and all mathematics, including transmathematics
- Conjecture: class theory with transarithmetic has no paradoxes!

Future Work

- Collaborate with set-theory specialists
- Develop a slipstreamed theorem prover, executing on a transcomputer
- Promote Transmathematics
- Google+ Community Transmathematica: https://plus.google.com/u/0/communities/103261551046378190173

Discussion

$$? \lor !$$