PERSPEX MACHINE IX: TRANSREAL ANALYSIS

Perspex Machine IX: Transreal Analysis

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Abstract

We introduce transreal analysis as a generalisation of real analysis. We find that the generalisation of the real exponential and logarithmic functions is well defined for all transreal numbers. Hence, we derive well defined values of all transreal powers of all non-negative transreal numbers. In particular, we find a well defined value for zero to the power of zero. We also note that the computation of products via the transreal logarithm is identical to the transreal product, as expected. We then generalise all of the common, real, trigonometric functions to transreal functions and show that transreal (sin x)/x is well defined everywhere. This raises the possibility that transreal analysis is total, in other words, that every function and every limit is everywhere well defined. If so, transreal analysis should be an adequate mathematical basis for analysing the perspex machine – a theoretical, super-Turing machine that operates on a total geometry. We go on to dispel all of the standard counter "proofs" that purport to show that division by zero is impossible. This is done simply by carrying the proof through in transreal arithmetic or transreal analysis. We find that either the supposed counter proof has no content or else that it supports the contention that division by zero is possible. The supposed counter proofs rely on extending the standard systems in arbitrary and inconsistent ways and then showing, tautologously, that the chosen extensions are not consistent. This shows only that the chosen extensions are inconsistent and does not bear on the question of whether division by zero is logically possible. By contrast, transreal arithmetic is total and consistent so it defeats any possible "straw man" argument. Finally, we show how to arrange that a function has finite or else unmeasurable (nullity) values, but no infinite values. This arithmetical arrangement might prove useful in mathematical physics because it outlaws naked singularities in all equations.

Keywords: transreal arithmetic, transreal analysis, singularities, unmeasurable quantities.

1. Introduction

Real arithmetic is extended to transreal arithmetic[3] by the inclusion of three, distinct numbers: $-\infty = -1/0$, $\Phi = 0/0$, and $\infty = 1/0$. The new arithmetic is axiomatised and shown to be consistent in.[4] Transreal arithmetic is a total arithmetic, which means that every arithmetical operation can be applied to any number such that the result is a number. This has consequences for real analysis. It means there are no undefined numbers nor undefined values of any function written as a well formed (or badly formed[3]) formula of arithmetic that has no free variables. In other words, any formula that has all of its variables instantiated can be evaluated arithmetically to give a well defined value. This comes as quite a surprise, which brings a new understanding of the relationship between arithmetic and analysis in its wake.

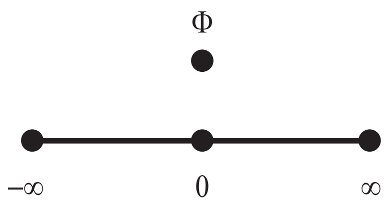

Figure 1: Transreal-number space consisting of the real-number line extended by two infinities at the extremes of the line and augmented with a number, nullity, lying off the line.

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The extended number-line is usually constructed by taking the real-number line and adding two objects: a limit at the positive extreme of the line and a limit at the negative extreme. A function may approach an *asymptotic infinity* as $\lim_{x \to y} f(x) \to \pm\infty$, but it cannot arrive exactly at an asymptotic infinity. Notice the use of an arrow, $\to$, to denote the approach to an asymptotic limit. Asymptotic limits at the extremes of the number line are useful in analysis, but play no part in standard arithmetic because they do not obey all of the axioms of standard arithmetic and so are not standard numbers. By contrast, $1/0 = \infty$ and $-1/0 = -\infty$ are *existential infinities* or *exact infinities*. They do obey all of the axioms of transreal arithmetic[4] so they are transreal numbers. A function $f(x) = \pm\infty$ evaluates exactly to positive or negative infinity and, in the limit, a function arrives exactly at the infinities as $\lim_{x \to y} f(x) = \pm\infty$. Notice the use of the equals sign, $=$, to indicate that the exact infinity is achieved, in contrast to the use of an arrow which denotes the approach to an asymptotic infinity. When we wish to distinguish a limit approached from below or above we write the limit, respectively, with a superscript minus sign or a superscript plus sign. With the limit approached from below we write $\lim_{x \to y^-} f(x) = \pm\infty$, that is $x \to y^-$, and, from above, we write $\lim_{x \to y^+} f(x) = \pm\infty$, that is $x \to y^+$.

The fact that we have access to the infinities as exact numbers makes a significant difference to the treatment of infinite series and their limits, but the existence of an exact nullity is even more profound. The number nullity was shown to lie off the number line by a geometrical construction[1] before its arithmetical properties were discovered. When nullity arises in a calculation it means that the formula just evaluated does not specify a unique solution on the real-number line extended by the infinities. This may be because there are no solutions at all on the extended number line, so that nullity is the "trivial" solution, or it may be that there is more than one solution. It is quite commonly the case that the transarithmetical evaluation of a series produces the number nullity, for example $\tanh(\infty) = \Phi$, whereas transreal analysis produces a different, specific, answer, here $\tanh(\infty) = 1$. This gives us a heuristic for developing transreal generalisations of real functions. We first evaluate them transarithmetically, but if real or transreal analysis suggests a different solution then we adopt the limit as a definition. Conversely, if real analysis gives no limit we define the limit to be nullity. No special limits need be adopted in the exponential or logarithmic functions when we write the exponential series correctly.

Transreal arithmetic defines division via the reciprocal $(a/b)^{-1} = b/a$, not via the multiplicative inverse $a \times a^{-1} = 1$. For example, we obtain the reciprocal of zero $0^{-1} = (0/1)^{-1} = 1/0 = \infty$ when there is no multiplicative inverse of zero. Indeed, $0 \times 0^{-1} = \frac{0}{1} \times \frac{1}{0} = \frac{0}{0} = \Phi$ and $\Phi \neq 1$ proves that the multiplicative inverse of zero does not exist in transreal arithmetic. One consequence of this is that functions defined via standard reciprocals or negative powers sometimes need to be written out explicitly. For example, with $-x < 0$ we define e^{-x} as $(e^x)^{-1}$, whence we find that the exponential function is well defined for all real and transreal arguments. This allows us to extend the logarithmic and trigonometric functions, though some trigonometric functions were extended earlier.[2] Having obtained the transreal logarithm we find that all powers of all non-negative numbers are well defined, in particular $0^0 = \Phi$. We also find that $1^{\pm\infty} = 1^{\Phi} = \Phi$. This distinguishes unity raised to a power of an infinity, $1^{\pm\infty} = \Phi$, from unity raised to a power approaching an infinity in the limit: $\lim_{x \to \infty} 1^x = 1$ and $\lim_{x \to -\infty} 1^x = 1$. This distinction is essential to the transreal, trigonometric identity $\cos^2 x + \sin^2 x = 1^x$, and to similar identities. Having obtained the transreal logarithm we find that all products obtained via logarithms are identical to the transreal product, as expected. This speaks to the consistency of the transreal exponential and logarithm, though it does not prove their consistency.

For $x > 0$ we define $\log(-x) = \Phi$. Following our heuristic of accepting more specific calculation paths in preference to paths leading to nullity, we may later define the logarithm of a negative, transreal number to be a transcomplex number. But as transreal arithmetic is total we must accept $\log(-x) = \Phi$ for the transreal logarithm. That is, whilst the real logarithm of a negative number is undefined, the transreal logarithm of a negative number is defined to be nullity. In both cases the logarithm can be extended to give complex solutions, though we do not describe the transcomplex numbers here.

On examining $(\sin x)/x$ we find that $(\sin 0)/0 = 0/0 = \Phi$. As nullity lies off the real number line it follows that $(\sin x)/x$ is discontinuous at zero. As usual, $\lim_{x \to 0^-} (\sin x)/x = 1$ and $\lim_{x \to 0^+} (\sin x)/x = 1$. Hence $(\sin x)/x$ is well defined everywhere. By evaluating the power series, we find that $\sin \pm\infty = \Phi = \sin\Phi$. And as the real limits $\lim_{x \to \pm\infty} \sin x$ do not exist we are free to define the transreal limits $(\lim_{x \to \pm\infty} \sin x) = \Phi$.

We also dissolve various counter "proofs" that purport to show that division by zero is impossible. Finally, we show how to arrange that a function has finite or else unmeasurable (nullity) values, but no infinite values. This arithmetical arrangement might prove useful in mathematical physics.

2. The Transreal Exponential Function

2.1 Definition

The transreal exponential function is defined as follows.

$$\exp(x) = \begin{cases} (\exp(-x))^{-1} : x < 0 \\ \lim_{k \to \infty} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} : \text{otherwise} \end{cases} \quad \text{[E 1]}$$

With this definition the transreal exponential of a real number is identical to the real exponential of the number. It remains only to find the exponential of the strictly transreal numbers.

$$\exp(\Phi) = \lim_{k \to \infty} 1 + \frac{\Phi}{1!} + \frac{\Phi^2}{2!} + \ldots + \frac{\Phi^k}{k!} = \lim_{k \to \infty} 1 + \frac{\Phi}{1!} + \frac{\Phi}{2!} + \ldots + \frac{\Phi}{k!} = \lim_{k \to \infty} 1 + \Phi + \Phi + \ldots + \Phi = \Phi \quad \text{[E 2]}$$

$$\exp(\infty) = \lim_{k \to \infty} 1 + \frac{\infty}{1!} + \frac{\infty^2}{2!} + \ldots + \frac{\infty^k}{k!} = \lim_{k \to \infty} 1 + \frac{\infty}{1!} + \frac{\infty}{2!} + \ldots + \frac{\infty}{k!} = \lim_{k \to \infty} 1 + \infty + \infty + \ldots + \infty = \infty \quad \text{[E 3]}$$

$$\exp(-\infty) = (\exp(\infty))^{-1} = \infty^{-1} = (1/0)^{-1} = 0/1 = 0 \quad \text{[E 4]}$$

Notice that [E 3] and [E 4] are in agreement with the real (unbounded) limits of the exponential, $\lim_{x \to \infty} \exp(x)$ and $\lim_{x \to -\infty} \exp(x)$, respectively. Hence, the transreal, exponential function is well defined everywhere.

2.2 Addition theorem

The real, exponential function obeys the addition theorem $\exp(x+y) = \exp(x) \times \exp(y)$. Hence, $\exp(x) = e^x$ for all real x. It remains to be shown that the addition theorem holds when any or all of x, y are strictly transreal. As transreal addition and multiplication are commutative[4] we need consider only x strictly transreal and y arbitrarily transreal. As usual, R^T, denotes the set of transreal numbers.[4]

Firstly, let $x = \Phi$ when $y \in R^T$. Then $\exp(x+y) = \exp(\Phi+y) = \exp(\Phi) = \Phi$ and $\exp(x) \times \exp(y) = \exp(\Phi) \times \exp(y) = \Phi \times k = \Phi$ for some $k \in R^T$, as required. Secondly, let $x = \infty$ when $y \geq 0$ then $\exp(\infty + y) = \exp(\infty) = \infty$ and $\exp(\infty) \times \exp(y) = \infty \times k = \infty$ for some $k \geq 1$, as required. Thirdly, let $x = \infty$ when $-\infty < y < 0$ then $\exp(x+y) = \exp(\infty+y) = \exp(\infty) = \infty$ and $\exp(x) \times \exp(y) = \exp(\infty) \times \exp(y) = \infty \times k = \infty$ for some $0 < k < 1$, as required. Fourthly, let $x = \infty$ when $y = -\infty$ then $\exp(x+y) = \exp(\infty - \infty) = \exp(\Phi) = \Phi$ and $\exp(x) \times \exp(y) = \exp(\infty) \times \exp(-\infty) = \infty \times 0 = \Phi$, as required. The cases with $x = -\infty$ are obtained similarly. Hence the addition theorem holds for all transreal numbers and we may write $\exp(x) = e^x$ for all transreal x.

2.3 Derivatives of the exponential

We adopt the standard definition of the derivative of the exponential: $\frac{de^x}{dx} = e^x$.

3. The Transreal, Natural Logarithm

3.1 Definition

We adopt the standard definition of the natural logarithm, $e^{\ln x} = \ln e^x = x$, but we define $\ln(-x) = \Phi$ when $-x < 0$ so as to keep the logarithm transreal. In future work we may drop this side condition so as to allow transcomplex numbers in the domain and range of the exponential and logarithmic functions.

The definition just adopted is identical to the real logarithm wherever it is defined. In addition, the following three logarithms, [E 5] to [E 7], are derived from [E 2] to [E 4] respectively; and [E 8] is the side condition, just given, to force the range of the transreal logarithm to be transreal rather than transcomplex.

$$\ln\Phi = \ln e^{\Phi} = \Phi \qquad \text{[E 5]}$$

$$\ln\infty = \ln e^{\infty} = \infty \qquad \text{[E 6]}$$

$$\ln 0 = \ln e^{-\infty} = -\infty \qquad \text{[E 7]}$$

$$\ln(-x) = \Phi : -x < 0 \qquad \text{[E 8]}$$

3.2 Products

Products of non-negative, transreal x, y may be computed via the transreal logarithm as $x \times y = e^{(\ln x + \ln y)}$. This is summarised in the following table. The table is consistent with the products obtained via the axioms of transreal arithmetic[4] or, for that matter, via real arithmetic. The table contains real numbers $0 < a < 1$ and $1 < b < \infty$.

$x \times y$		y: 0	a	1	b	∞	Φ
x	0	0	0	0	0	Φ	Φ
	a	0	a^2	a	ab	∞	Φ
	1	0	a	1	b	∞	Φ
	b	0	ab	b	b^2	∞	Φ
	∞	Φ	∞	∞	∞	∞	Φ
	Φ	Φ	Φ	Φ	Φ	Φ	Φ

Table 1: Multiplication via the logarithm is consistent with the real and transreal product.

3.3 Powers

A non-negative, transreal number, x, raised to the transreal power, y, may be computed via the transreal logarithm as $x^y = e^{y \ln x}$. This is summarised in the following table. The table is consistent with the powers that can be obtained via the axioms of transreal arithmetic[4] or, for that matter, via real arithmetic. The table contains real numbers $0 < a < 1$ and $1 < b < \infty$.

x^y		y									
		$-\infty$	$-b$	-1	$-a$	0	a	1	b	∞	Φ
x	0	∞	∞	∞	∞	Φ	0	0	0	0	Φ
	a	∞	a^{-b}	a^{-1}	a^{-a}	1	a^a	a	a^b	0	Φ
	1	Φ	1	1	1	1	1	1	1	Φ	Φ
	b	0	b^{-b}	b^{-1}	b^{-a}	1	b^a	b	b^b	∞	Φ
	∞	0	0	0	0	Φ	∞	∞	∞	∞	Φ
	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ

Table 2: Powers obtained via the logarithm are consistent with real and transreal powers.

It is worthy of note that 0^0 is well defined. Specifically: $0^0 = e^{0 \ln 0} = e^{0 \times (-\infty)} = e^{\frac{0}{1} \times \frac{-1}{0}} = e^{\frac{0}{0}} = e^{\Phi} = \Phi$. Perhaps the only surprises in the table are $1^{\infty} = e^{\infty \ln 1} = e^{\infty \times 0} = e^{\frac{1}{0} \times \frac{0}{1}} = e^{\frac{0}{0}} = e^{\Phi} = \Phi$ and $1^{-\infty} = e^{-\infty \ln 1} = e^{-\infty \times 0} = e^{\frac{-1}{0} \times \frac{0}{1}} = e^{\frac{0}{0}} = e^{\Phi} = \Phi$. This distinguishes unity raised to a power of infinity, $1^{\pm\infty} = \Phi$, from unity raised to a power approaching infinity in the limit: $\lim_{x \to \infty} 1^x = 1$ and $\lim_{x \to -\infty} 1^x = 1$. This distinction is essential to the transreal, trigonometric identity $\cos^2 x + \sin^2 x = 1^x$, and to similar identities, thereby dissolving any surprise that may have been felt on first acquaintance with the identity $1^{-\infty} = 1^{\infty} = 1^{\Phi} = \Phi$.

Recall, Figure 1, that nullity lies off the number line so any function $f(x) = \Phi$ is discontinuous at x. Consider the function $f(x) = 0^x$ shown in the first row of the table above. This function is nullity off the line, $f(x) = \Phi : x = \Phi$, constant below zero, $f(x) = \infty : x < 0$, discontinuous at zero, $f(x) = \Phi : x = 0$, and constant above zero, $f(x) = 0 : x > 0$. Thus, the function is nullity at the point, here $x = 0$, where it jumps discontinuously from infinity to zero. This behaviour is also observed in the fifth row ($x = \infty$), and the first ($y = -\infty$) and ninth ($y = \infty$) columns. In the final row ($x = \Phi$) and column ($y = \Phi$) the function is everywhere nullity. In all other rows and columns the functions pass smoothly through unity.

4. Addition and Multiplication of Exponents

The standard theorem for the addition of real exponents of positive real numbers is:

$$a^{x+y} = a^x \times a^y : a > 0 \text{ where } a, x, y \in R \qquad \text{[E 9]}$$

Whilst the theorem is true, the guarding clause is too restrictive, even for the real numbers. For example, taking $a = 0$ when $x, y = 1$ gives $0^{1+1} = 0^2 = 0 \times 0 = 0 = 0 \times 0 = 0^1 \times 0^1$. And taking a, x, y transreal we find that many more of the cases excluded by the standard guarding clause are true. For example, taking all of $a, x, y = 0$ we have the equality $0^{0+0} = 0^0 = \Phi = \Phi \times \Phi = 0^0 \times 0^0$. On examining cases we find that the following less restrictive guarding clause holds, giving a more general theorem for the addition of transreal exponents of a non-negative transreal number.

$$a^{x+y} = a^x \times a^y : \neg(a < 0) \wedge (\neg((a = 0) \wedge (x + y \neq 0) \wedge ((x = 0) \vee (y = 0)))) \text{ where } a, x, y \in R^T \qquad \text{[E 10]}$$

The standard theorem for the multiplication of real exponents of positive real numbers is:

$$a^{x \times y} = (a^x)^y : a > 0 \text{ where } a, x, y \in R \qquad \text{[E 11]}$$

As before, the guarding clause is too restrictive, even for the real numbers. For example, taking $a = 0$ when $x, y = 1$ gives $0^{1 \times 1} = 0^1 = 0 = 0^1 = (0^1)^1$. And taking a, x, y transreal we find that many more of the cases, excluded by the standard guarding clause, are true. For example, taking all of $a, x, y = 0$ we have the equality $0^{0 \times 0} = 0^0 = \Phi = \Phi^0 = (0^0)^0$. On examining cases we find that the following less restrictive guarding clause holds, giving a more general theorem for the multiplication of transreal exponents of a non-negative transreal number.

$$a^{x \times y} = (a^x)^y : \neg(a < 0) \text{ where } a, x, y \in R^T \qquad \text{[E 12]}$$

5. The Transreal Trigonometric Functions

The real, trigonometric functions, $f(x)$, can all be evaluated as power series of real x, but $\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1-1}{0} = \frac{0}{0} = \Phi$ so any power series with alternating signs gives $f(\pm\infty) = \Phi$. We accept nullity unless there is good reason not to. For example, we accept a limit where it exists. Given $\lim_{x \to \infty} \tanh x = 1$ and $\lim_{x \to -\infty} \tanh x = -1$ we accept $\tanh \pm\infty = \pm 1$ and, conversely, $\text{arctanh}(\pm 1) = \pm\infty$. We accept an infinity when the function is unbounded. For example, $\tan(\pm\pi/2) = \pm\infty$. But were there is no limit and no unbounded value, because the function is cyclic, we accept the value nullity obtained by evaluating the series expansion. For example, $\tan(\pm\infty) = \Phi$. This does not preclude the possibility that some other good reason might be found to supply a non-nullity value to a trigonometric function, but we do not know of any such cases.

All trigonometric functions give $f(\Phi) = \Phi$ so we augment the principal range of the arc functions with nullity. For example: $\text{arctanh}\Phi = \Phi$, $\text{arctanh}(\pm\infty) = \pm\pi/2$, and $-\pi/2 < \text{arctanh}x < \pi/2 : -\infty < x < \infty$. Similarly $\arcsin\Phi = \arcsin(\pm\infty) = \Phi$ and $-\pi/2 \le \arcsin x \le \pi/2 : -1 \le x \le 1$.

We define some of the transreal, trigonometric functions next, but a fuller treatment must await the development of the transcomplex numbers and a fuller analysis of the trigonometric identities. The hyperbolics are defined in terms of the transreal exponential.

5.1 Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{[E 13]}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{[E 14]}$$

$$\tanh x = \begin{cases} -1 : x = -\infty \\ 1 : x = \infty \\ \dfrac{e^x - e^{-x}}{e^x + e^{-x}} : \text{otherwise} \end{cases} \qquad \text{[E 15]}$$

$$\coth x = \begin{cases} -1 : x = -\infty \\ \Phi : x = 0 \\ 1 : x = \infty \\ \dfrac{e^x + e^{-x}}{e^x - e^{-x}} : \text{otherwise} \end{cases} \qquad \text{[E 16]}$$

Notice that the hyperbolic tangent is continuous on the extended number line, $-\infty \le x \le \infty$, but the hyperbolic co-tangent is discontinuous at $\coth 0 = \Phi$, though it does have limits approaching zero in the neighbourhood of the discontinuity: $\lim_{x \to 0^-} \coth x = -\infty$ and $\lim_{x \to 0^+} \coth x = \infty$.

The real sine, cosine, tangent, co-tangent, secant, and co-secant functions are all cyclic so they all have the behaviour $f(\pm\infty) = f(\Phi) = \Phi$ and are otherwise identical to the real functions with the asymptotic infinities replaced by exact infinities. Of these functions, only the tangent and co-tangent are discontinuous on the extended number line so we define the following discontinuities and limits in the region of the discontinuities.

$$\tan(k\pi + \pi/2) = \Phi : k \in Z \qquad \text{[E 17]}$$

$$\lim_{x \to (k\pi + \pi/2)^-} \tan x = \infty \qquad \text{[E 18]}$$

$$\lim_{x \to (k\pi + \pi/2)^+} \tan x = -\infty \qquad \text{[E 19]}$$

$$\cot(k\pi) = \Phi : k \in Z \qquad \text{[E 20]}$$

$$\lim_{x \to (k\pi)^-} \cot x = -\infty \qquad \text{[E 21]}$$

$$\lim_{x \to (k\pi)^+} \cot x = \infty \qquad \text{[E 22]}$$

5.2 Identities

Many of the real, trigonometric identities can be extended to transreal, trigonometric identities by incorporating the number 1^x. We have $1^x = 1 : x \in R$ and $1^x = \Phi : x \in \{-\infty, \infty, \Phi\}$. See Section 3.3 above. The number 1^x is used to distinguish the real solutions from the "no information" solution Φ. For example, consider the following identity.

$$\cos^2 x + \sin^2 x = 1^x \qquad \text{[E 23]}$$

When $x \in R$ we have $\cos^2 x + \sin^2 x = 1$, as usual. Otherwise we have the cyclic cases $\cos^2(\pm\infty) + \sin^2(\pm\infty) = \Phi + \Phi = \Phi$ and we have the discontinuous case $\cos^2(\Phi) + \sin^2(\Phi) = \Phi + \Phi = \Phi$. Equation [E 23] encompasses all of these cases.

We may solve [E 23] to obtain relationships between the real sine and cosine. This gives us two solutions depending on whether we carry $1^x \in R : x \in R$ into the solution or whether we substitute the real solution $1 = 1^x : x \in R$. For example, $\sin x = \sqrt{1^x - \cos^2 x}$ and $\sin x = \sqrt{1 - \cos^2 x}$ are both true for all transreal x.

Similarly:

$$\cosh^2 x - \sinh^2 x = 1^x \qquad \text{[E 24]}$$

Solving for the real cases we obtain both $\sinh x = \sqrt{\cosh^2 x - 1^x}$ and $\sinh x = \sqrt{\cosh^2 x - 1}$, but on examining the strictly transreal cases we find that the first of these identities is false, whilst the second is everywhere true. For a false case consider $\infty = \sinh\infty = \sqrt{\cosh^2\infty - 1^\infty} = \sqrt{\infty - \Phi} = \sqrt{\Phi} = \Phi$.

This phenomenon of deriving several solutions, only some of which are valid, is a normal part of algebra. Whenever one derives a solution set to some equation it is advisable to test the existence of the solutions.

On checking cases we find that very many of the standard trigonometric identities either hold for all transreal numbers or else can be modified so that the do hold by incorporating the number 1^x.

6. Dissolving Counter Proofs

There are a number of counter proofs of the possibility of division by zero in arithmetic. All such counter proofs are falsified or rendered irrelevant by the machine proof of consistency of transreal arithmetic.[4] Nonetheless, it is useful to examine these counter proofs to understand the objections to division by zero that have been held for so very long.

6.1 Counter Proof: Fields

All algebraic counter proofs, that the author has seen, rely on defining division via the multiplicative inverse, $a \div b = ab^{-1} : bb^{-1} = 1$, whereas transreal arithmetic defines division via the reciprocal, $\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b}\right)\left(\frac{c}{d}\right)^{-1} : \left(\frac{c}{d}\right)^{-1} = \left(\frac{d}{c}\right)$. Here an irrational number x is rendered as $x/1$. Thus, the algebraic counter proofs simply do not apply to transreal arithmetic because they use a different definition of division. For example, the well known proof that a field with more than one element has no multiplicative inverse of zero is simply a theorem about the multiplicative inverse and has no bearing on division.

6.2 Counter Proof: Sin(x)/x

There are counter proofs in analysis, but these rely on properties that do more harm to analysis than the alternative of allowing division by zero. Quite commonly the counter "proofs" seek to impose continuity across a discontinuous function. Here is one purported counter proof. (The author issued a challenge, in the United Kingdom and the United States, claiming that there are no valid counter proofs to division be zero in arithmetic. He received this candid reply, amongst others, which is slightly edited and reported anonymously.)

> *If division by zero is allowed then one loses the mathematical idea of limits. For example, consider the function $f(x) = (\sin x)/x$, defined for non-zero x. As x tends to zero, but is not equal to zero, $f(x)$ can be shown to tend to unity. Therefore, if one wishes to extend $f(x)$ to include the value $x = 0$, the natural thing to do would be to define $f(0) = 1$. Such extensions are commonplace in mathematics, particularly in analysis.*
>
> *Now consider the slightly different function $g(x) = (\sin 2x)/x$ defined for non-zero x. This time, as x tends to zero, $g(x)$ tends to two. Again, one naturally extends $g(x)$ to include $x = 0$ by defining $g(0) = 2$. Thus, by keeping $(\sin kx)/x = 0/0$ undefined, for all real k, we retain the freedom to supply only the limits that are needed. But if $0/0$ is a fixed number then we cannot do this and the limit of $(\sin kx)/x$ at $x = 0$ does not exist, nor do many of the limits obtained in analysis. Allowing $0/0$ to be a fixed value would destroy the whole of calculus, differential equations, and mathematical physics.*

The respondent is to be commended for his candour, but there is an alternative to his "natural" choices. We have $(\sin x)/x = \Phi : x \in \{0, -\infty, \infty, \Phi\}$ so that $(\sin x)/x$ is discontinuous at $x = 0$ and has the usual limits: $\lim_{x \to 0^-} (\sin x)/x = 1$ and $\lim_{x \to 0^+} (\sin x)/x = 1$. Hence, all functions $(\sin kx)/x = 0/0$ are well defined everywhere and have well defined limits at $x = 0$ (and also $x = \pm\infty, \Phi$). It is not necessary to leave $0/0$ undefined. Indeed, allowing it to take on arbitrary values in order to make arbitrary functions continuous would seem to inhibit the whole of calculus, differential equations, and mathematical physics, whereas accepting $0/0$ as a well defined number would seem to make the whole of calculus, differential equations, and mathematical physics total, and therefore more secure. By accepting that $0/0$ is a fixed number we remove arbitrary extensions and, as our respondent said, "*Such extensions are commonplace in mathematics, particularly in analysis,"* so we do a very great deal of good by removing these harmful extensions.

6.3 Benefit: Harmless Singularities

One good that we might do in mathematical physics is to limit the harm done by singularities. We may allow that some physical functions go to infinity at a singularity, but in many cases the physical influence of these infinite values is felt via a multiplicative function involving zero. The product of the infinity and the zero is nullity, which lies off the number line, which, therefore, has no influence on any part of the number line. In this way, the influence of the infinite value at the singularity is excised from the space of real numbers and, we suppose, is excised from the observable, physical universe.

More generally, we may arrange that a function has finite, non-zero, values or else unmeasurable, nullity, values, but no infinite values. For example, let $f(x) = x^2/x$. Then $f(x) = \Phi$ when $x \in \{-\infty, 0, \infty, \Phi\}$. Otherwise $f(x)$ is non-zero and finite. Most generally, we may arrange that a function has arbitrary finite values or nullity. For example, let $g(x) = 1^x \times x$ then $g(x) = \Phi$ when $x \in \{-\infty, \infty, \Phi\}$. Otherwise $g(x)$ is finite. That is, $-\infty < g(x) < \infty$.

These examples are somewhat artificial, but we have considerable freedom to construct such functions, and may replace any term x with some $f(x)$ so that all physical models give finite or unmeasurable values, but no infinite ones. This removes naked singularities from all equations. It is an open question whether there are physical process where nullity arises naturally in the mathematical description of the process. An obvious place to look would be physically unmeasurable quantities that arise in the quantum wave equation and in all cases of singularities. If any physical example of nullity is found, this would be another good reason to prefer transreal arithmetic over real arithmetic.

6.4 Counter Proof: f(x) = 1/x

The following argument is an edited version of an argument appearing in Wikipedia.[5]

> *Consider the function $f(x) = 1/x$. As x approaches zero from below $f(0^-) \to -\infty$ and as x approaches zero from above $f(0^+) \to \infty$. But the choice of $f(0) = -\infty$ or $f(0) = \infty$ is arbitrary and we cannot have both, so there is no well defined value of the function $f(x) = 1/x$ at $x = 0$.*

But this is argument is predicated on the assumption that $f(x)$ is a function. If, instead, $f(x)$ is a mapping we may have the harmless definition:

$$f(x) = \begin{cases} \{-\infty, \infty\} : x = 0 \\ 1/x : \text{otherwise} \end{cases}$$

This counter "proof," like so many others in analysis, relies on an unstated assumption of continuity where no such assumption is warranted. It would be interesting to speculate why authors and readers do not notice this assumption. But let us allow the assumption. Then $f(0) = 1/0 = \infty$. Similarly, in $g(x) = -1/x$ we have $g(0) = -1/0 = -\infty$. In each case the choice of $\pm\infty$ is fixed, it is not arbitrary, it depends entirely on the choice of sign in ± 1. The argument from limits is irrelevant. Again, it would be interesting to speculate why authors and readers do not notice the unstated assumption of continuity or the unwarranted appeal to limits. The value of infinity is fixed by arithmetic, not by analysis.

6.5 Counter Proof: Infinite Ratios

The following argument is an edited version of an argument appearing in Wikipedia.[5]

> *Assuming that several natural properties of the real numbers extend to the infinities we have* $+\infty = 1/0 = 1/(-0) = -1/0 = -\infty$ *so that* $+\infty$ *and* $-\infty$ *are identical. Hence, we must adopt an unsigned infinity.*

Adopting an unsigned infinity is harmless, though it leads to a different algebraic structure than that discussed here. Such an infinity occurs, for example, in projective geometry. But, in any case, the argument rests on unstated assumptions. What are the several natural properties of the real numbers that extend to the infinities? And why should we use these incoherent extensions rather than coherent ones? Carrying out the evaluation in transreal arithmetic[4] produces the unproblematic tautology: $+\infty = 1/0 = 1/(-0) = 1/0 = +\infty$.

6.6 Counter Proof: Fallacies Obtained on Dividing by Zero

The following argument is an edited version of an argument appearing in Wikipedia.[5]

> *Consider the formal (syntactic) equality* $x^2 - x^2 = x^2 - x^2$. *Factoring differently on both sides yields* $(x-x)(x+x) = x(x-x)$. *Dividing both sides by* $x-x$ *gives* $(0/0)(x+x) = x(0/0)$. *Simplifying yields* $1(x+x) = x(1)$ *whence* $2x = x$, *but on substituting non-zero values for* x *we obtain a contradiction. Therefore division by zero is impossible.*

The unstated assumption here is that $0/0 = 1$, but this is false in transreal arithmetic.[4] Taking up the argument from $(0/0)(x+x) = x(0/0)$ we multiply out giving $(0/0) = (0/0)$ which is trivially (formally) true.

6.7 Counter Proof: Zero to the Power of Zero

There are various demonstrations that if $0^0 \in R$ then both real arithmetic and real analysis are inconsistent. We have already shown, by considering the logarithm, that $0^0 = \Phi \notin R$, but it is useful to carry out the evaluation using exponents:

$$0^0 = 0^{(1-1)} = 0^1 \times 0^{-1} = \left(\frac{0}{1}\right)^1 \times \left(\frac{0}{1}\right)^{-1} = \frac{0}{1} \times \frac{1}{0} = \frac{0}{0} = \Phi$$

The reader is asked to consider what makes this evaluation so terribly difficult that it has not been obtained earlier in the 1,200 years since the invention of zero? Is it the last part of the equation, defining nullity, $0/0 = \Phi$, that is so difficult? If so, why does this difficulty persists when the modern reader has experience of definitions such as $\sqrt{-1} = i$?

7. The Perspex Machine

The perspex machine[3] is a super-Turing machine that operates geometrically in a space with transreal co-ordinates. Analysing it from first principles, without access to transreal analysis or transreal algebra, is extremely cumbersome. The techniques developed here might assist in the analysis of the machine and might be more generally useful.

The perspex machine uses the single computer instruction:

$$\text{continuum}(\overrightarrow{z+p}) + \sum_i \left(\overrightarrow{x^{(i)} + p^{(i)}}\right)\left(\overrightarrow{y^{(i)} + p^{(i)}}\right) \rightarrow (\overrightarrow{z+p});$$

$$\text{jump}((\overrightarrow{z+p})_{11}, t).$$

Without examining this in detail, we can see that the first line of the formula is a general-linear transformation. This raises the possibility that the perspex machine can be implemented as a wave machine with computational waves being superimposed on each other so that every perspex in the machine is simultaneously engaged in many computations. Furthermore, if control is transmitted in a wave front that lags the transmission of data in a wave front then data will always be present when control arrives. Such a synchronous perspex machine will operate as an unstallable pipeline, giving maximum performance at all times and places in the machine. Such an asynchronous perspex machine will also be unstallable, but it will be delayed by the slower transmission of computational waves through parts of the machine or, conversely, will be accelerated by the relatively faster transmission of computational waves through parts of the machine. It might be that if a perspex wave machine is theoretically possible, it can be implemented as a quantum computer. These conjectures may be examined or otherwise developed in future work.

8. Conclusion

Transreal arithmetic is total. This changes the relationship between arithmetic and analysis, because transreal arithmetic can deal with the infinities independently of analysis. In this circumstance the task is to adopt definitions, where necessary, to make the arithmetical value and the limit consistent. We find that no special definitions are needed in the exponential and the logarithm, beyond writing out a negative exponent as an explicit reciprocal, though special definitions are needed in some hyperbolic, trigonometric equations.

We dispose of various purported counter proofs which aim to show that division by zero is inconsistent. Many of these counter "proofs" make an appeal to limits, but the counter "proofs" are begged by some or all of the unstated assumptions: (i) the mapping in question is a function; (ii) the function is continuous; (iii) the choice of whether infinity is positive or negative is arbitrary. The last of these, self-defeating, assumptions deserves closer attention. Consider the set of natural numbers composed of the element zero and its successors: $N = \{0, 1, 2, \ldots\}$. We define the set of extended natural numbers $N^* = N \cup \{\infty\}$ such that $\infty > n : n \in N$. And we define the set of transnatural numbers $N^T = N^* \cup \{\Phi\} = N \cup \{\infty, \Phi\}$ such that $\Phi \neq n : n \in N^*$. On introducing the operation of subtraction, we observe that $0 = -0$ and we partition zero into a singleton set. Similarly, we observe[4] that $\Phi = -\Phi$ and we partition nullity into a singleton set. We call all of the remaining numbers the set of positive, transnatural numbers: $P = \{1, 2, \ldots, \infty\}$. We then go on to construct[4] the set of transintegers, Z^T, and observe that subtraction is closed on this set. We then construct[4] the set of transrational numbers, Q^T, and observe that division is closed on this set. Finally, we construct[4] the set of transreal numbers R^T and note that the operation of forming the sum of any series is closed on this set. At no point in this construction is there an opportunity to take ∞ negative. What alternative construction of the transreal numbers could the authors of the counter "proofs" have in mind when they claim that the choice of whether infinity is positive or negative is arbitrary? And if they had such a construction, why did they not give it? And as they did not give any such construction, what credence can we be expected to lend, however fleetingly, to their, purported, counter proofs?

We find that very many of the standard, trigonometric identities either hold for all transreal numbers or else can be modified so that the do hold by incorporating the number 1^x. In particular, we find that $\cos^2 x + \sin^2 x = 1^x$ and $\cosh^2 x - \sinh^2 x = 1^x$ for all $x \in R \cup \{-\infty, \infty, \Phi\}$. We conjecture that all real, trigonometric identities can be generalised to transreal, trigonometric identities by incorporating a term 1^x.

We show how to arrange that a function has finite or else unmeasurable (nullity) values, but no infinite values, thereby abolishing naked singularities from all equations. One such technique is to replace all terms x with x^2/x. This substitution ensures that $f(x) = x^2/x \neq 0$ and may be used to outlaw zero radii at a singularity. It is important to note that where equations describing some physical system naturally incorporate terms of the form x^2/x, cancelling x to give $f(x) = x$ *introduces* a singularity where no singularity existed before the arithmetical simplification. It is interesting to speculate how many singularities occurring in physical equations are a side effect of the failure to cancel terms correctly. More generally, substituting x with $1^x \times x$ ensures that $f(x) = 1^x \times x$ can take on any finite value, $-\infty < f(x) < \infty$, or nullity. Thus, arithmetical properties abolish singularities and explain how some physical quantities can be unmeasurable. Nullity lies off the number line and is incommensurate with every number on the line, so any physical quantity with value nullity is incommensurate with any real-number; but we suppose that all physically measurable quantities are commensurate with the real numbers, which is to state the hypothesis that all nullity values are physically unmeasurable. If a conservation law is added to a multivariate function then the function's value may convert from a measurable, real, value to an unmeasurable, nullity, value and back again, without ever involving an infinite value. Most importantly, this arithmetical arrangement does not require any special functions (such as quantum wave functions) to be introduced that define unmeasurability. Unmeasurability is an arithmetical property of nullity and may be expected to arise naturally in various physical systems. To the extent that transreal arithmetic performs a useful service by outlawing singularities and explaining physical unmeasurability, it is to be preferred to real arithmetic on physical grounds. Naturally, transreal arithmetic is to be preferred to real arithmetic on the grounds of arithmetical totality.

The concept of a limit is carried into sets of transreal numbers, X, by the definition $\sup(\{\Phi\} \cup X) = \sup\{X\}$. It follows that the whole of real analysis can be generalised to transreal analysis.

The number 0^0 is undefined in standard arithmetic, but in transreal arithmetic we have:

$$0^0 = 0^{(1-1)} = 0^1 \times 0^{-1} = \left(\frac{0}{1}\right)^1 \times \left(\frac{0}{1}\right)^{-1} = \frac{0}{1} \times \frac{1}{0} = \frac{0}{0} = \Phi$$

The reader is asked to consider what makes this evaluation so terribly difficult that it has not been obtained earlier in the 1,200 years since the invention of zero? Is it the last part of the equation, defining nullity, $0/0 = \Phi$, that is so difficult? If so, why does this difficulty persists when the modern reader has experience of definitions such as $\sqrt{-1} = i$?

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10. References

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