The Perspex Machine

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Papers

This presentation incorporates material from three papers.

- *Perspex Machine V:*
  Compilation of C Programs
  Matthew P. Spanner & James A.D.W. Anderson.

- *Perspex Machine VI:*
  A Graphical User Interface to the Perspex Machine
  Christopher J.A. Kershaw & James A.D.W. Anderson

- *Perspex Machine VII:*
  The Universal Perspex Machine
  James A.D.W. Anderson

Introduction

- The Perspex Machine unifies the Turing Machine with geometry so that any symbolic computation can be performed geometrically, though some geometrical computations have no symbolic counterpart.
- Even when simulated approximately on a digital computer, the Turing computable, geometrical properties of the perspex machine are useful.

Introduction

Practical simulations of the Perspex Machine use co-ordinates expressed in transrational numbers.

- Transrational arithmetic is a total arithmetic that illustrates a serious omission in mathematics and two corner-case errors in IEEE, floating-point arithmetic.
- All decimal expansions that can be computed by a Turing machine that halts are transrational numbers. In general, sequences of transrational numbers with less than quadratic convergence are non-monotonic. This has *physical* consequences.
- A compiler has been implemented that compiles a subset of C into perspex programs.

How Numbers are Defined

Traditionally, numbers are defined:

- as the solution set of an equation;
- as the result of an operation;
- by multiplication tables;
- by axioms.

Numbers as Solutions

- The natural numbers, N, are 1, 2, 3, …
- With $a, b \in N$ the equation $a + x = b$ has non-natural solutions $x = 0$ and $-x \in N$. Today, zero and the negative integers are considered to be numbers because they are solutions to this equation. The full solution set is the set of integers $Z = \{x : -x \in N\} \cup \{0\} \cup N$.
- Similarly, with $a, b \in Z$ and $a \neq 0$ the equation $a \times x = b$ has some non-integer solutions written as the fractions $x = b/a$ with $a \neq 0$. Today, these fractions, reduced to canonical form, are considered to be numbers because they are solutions to this equation. The full solution set is the set of rational numbers Q.

Numbers as Solutions

- But, with $a, b \in Z$ the equation $x = b/a$ has non-rational solutions when $a = 0$. These fractions *should* be considered to be numbers because they are solutions to this equation. When these fractions are reduced to canonical form they give rise to the transrational numbers Q^*.

- Challenge – either accept that fractions $b/0$ are numbers or else prove that they are not numbers.

Numbers via Operations

Making three changes to the operations of rational arithmetic gives rise to transrational arithmetic. Here $a, b \in Z$:

- $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$;
- $\frac{a}{b} = \frac{-a}{|b|}$ when $b < 0$;
- $-\frac{1}{|b|} < \frac{1}{|b|}$.

Numbers via Operations

For $k \in R$, $k > 0$ we define:

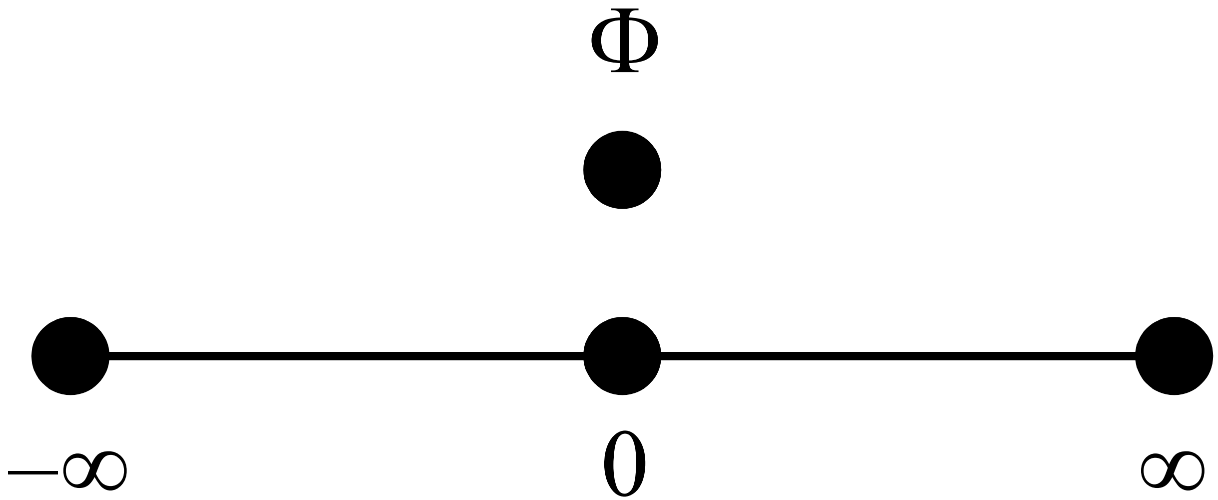
- infinity as $\infty = k/0 = 1/0$ with canonical form $1/0$;
- minus infinity as $-\infty = -k/0 = -1/0$ with canonical form $-1/0$;
- nullity as $\Phi = 0/0$.

Hence, $-\infty$, ∞ and Φ are both transrational and transreal numbers.

They are said to be *strictly* transrational and *strictly* transreal because they are not rational and not real.

Numbers via Operations

The infinities lie at the extremes of the number line and nullity lies off the number line.



Corner Case One

IEEE floating-point arithmetic uses only one sign bit, but two sign bits are needed to encode the sign of a number a.

$$\operatorname{sgn}(a) = \begin{cases} -1 : a < 0; \\ 1 : a > 0; \\ 0 : a = 0; \\ \Phi : a = \Phi. \end{cases}$$

Multiplication Tables

In the multiplication tables:

- $n_i, d_i \in Z$ and $n_i, d_i > 0$;
- **T** means unconditionally **T**rue;
- **F** means unconditionally **F**alse;
- **C** means **C**onditionally True or False identically as it is True or False in rational arithmetic.
- $\pm q_3$ is $+q_3$ or else $-q_3$ identically as in rational arithmetic.

Equality Table

$\frac{n_1}{d_1} = \frac{n_2}{d_2}$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	C	F	F	F	F	F
0	$0/1$	F	T	F	F	F	F
q_1	n_1/d_1	F	F	C	F	F	F
$-\infty$	$(-1)/0$	F	F	F	T	F	F
∞	$1/0$	F	F	F	F	T	F
Φ	$0/0$	F	F	F	F	F	T

Corner Case Two

IEEE floating-point arithmetic has:

$$\mathrm{NaN} \neq \mathrm{NaN}.$$

Whence:

$$\frac{0}{0} \neq \frac{0}{0}.$$

But transrational and transreal arithmetic have:

$$\frac{0}{0} = \frac{0}{0}.$$

Greater-Than Table

$\frac{n_1}{d_1} > \frac{n_2}{d_2}$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	C	F	F	T	F	F
0	$0/1$	T	F	F	T	F	F
q_1	n_1/d_1	T	T	C	T	F	F
$-\infty$	$(-1)/0$	F	F	F	F	F	F
∞	$1/0$	T	T	T	T	F	F
Φ	$0/0$	F	F	F	F	F	F

Addition Table

$\frac{n_1}{d_1} + \frac{n_2}{d_2}$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	$-q_3$	$-q_1$	$\pm q_3$	$-\infty$	∞	Φ
0	$0/1$	$-q_2$	0	q_2	$-\infty$	∞	Φ
q_1	n_1/d_1	$\pm q_3$	q_1	q_3	$-\infty$	∞	Φ
$-\infty$	$(-1)/0$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	Φ	Φ
∞	$1/0$	∞	∞	∞	Φ	∞	Φ
Φ	$0/0$	Φ	Φ	Φ	Φ	Φ	Φ

Multiplication Table

$\frac{n_1}{d_1} \times \frac{n_2}{d_2}$		$-q_2$	0	q_2	$-\infty$	∞	Φ
		$(-n_2)/d_2$	$0/1$	n_2/d_2	$(-1)/0$	$1/0$	$0/0$
$-q_1$	$(-n_1)/d_1$	q_3	0	$-q_3$	∞	$-\infty$	Φ
0	$0/1$	0	0	0	Φ	Φ	Φ
q_1	n_1/d_1	$-q_3$	0	q_3	$-\infty$	∞	Φ
$-\infty$	$(-1)/0$	∞	Φ	$-\infty$	∞	$-\infty$	Φ
∞	$1/0$	$-\infty$	Φ	∞	$-\infty$	∞	Φ
Φ	$0/0$	Φ	Φ	Φ	Φ	Φ	Φ

Challenge

Either:

- accept that fractions $b/0$, for all $b \in Z$, are numbers
- or else prove that they are not numbers.

The force of this challenge is that you should accept the numbers $-\infty$, ∞, Φ.

Walnut Cake Theorem

- The Walnut Cake Theorem has less strong pre-conditions and, hence, is more general than the Chinese Remainder Theorem.

- If a value is bounded on one side to a precision of $1/a$ and is bounded on the other side to a different precision $1/b$ then the value is bounded up to a precision of $1/ab$.

- In general there are many precisions $1/c$ with $a, b < c < ab$ that are less tight than the tightest of the bounds at the lesser precisions $1/a$ and $1/b$. The proof of this statement is in the theorem.

Walnut Cake Theorem

- In other words, in general, Turing computable decimal expansions with a convergence of less than $1/ab$, i.e. less than quadratic (O $1/b^2$ with $a < b$) are non-monotonic.

This is a surprise for people who believe that the calculus of continuous functions provides a sufficient model of digital arithmetic.

Walnut Cake Theorem

- A notable exception is binary arithmetic which has a low density of numbers on the number line.
- A large, but perverse, class of exceptions is the arithmetics where the base of successive digits grows quadratically! These arithmetics have a lower density than binary arithmetic.
- The remaining exceptions are the serendipitous compuations that always compute the tightest bound at every digit.
- Taking all of this together, we say that, in general, any arithmetic with a sufficiently high density computes sub-quadratic convergence non-monotonically.

Walnut Cake Theorem

Practical examples of non-monotonic performance include:

- sub-quadratic numerical algorithms;
- phenotypes generated from DNA;
- sub-quadratic symbolic computations performed on a perspex machine;
- scientific theories expressed in language.

It is astonishing that paradigm shifts in the scientific literature, and any literary enterprise, are a consequence of the properties of (trans)rational numbers.

Compiling C into Perspexes

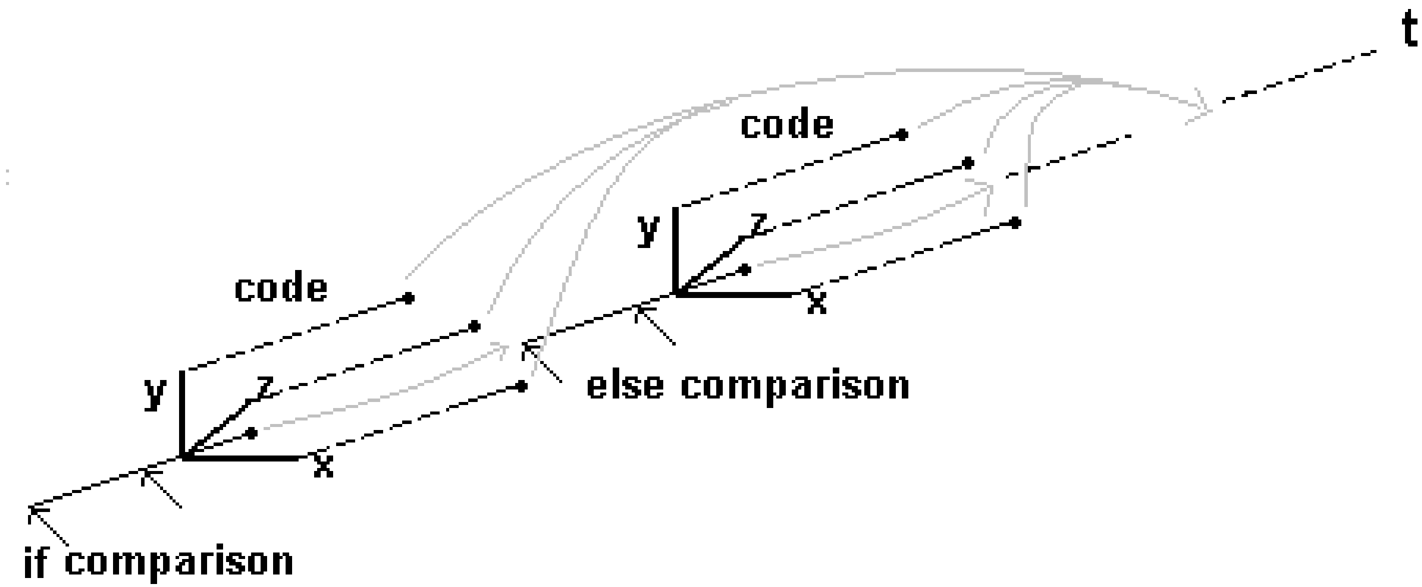
- A C to perspex compiler converts C source code into perspexes.
- Perspexes are geometrical transformations.
- The compiler uses perspective transformations, but the universal perspex machine uses general linear transformations.
- Source code of the Travelling Salesman problem.
- Still of the equivalent transformations.
- Movie of the equivalent transformations.

Compiling C into Perspexes

- The compiler operates by generating templates of geometrical transformations rather than templates of assembler or von Neumann machine code.

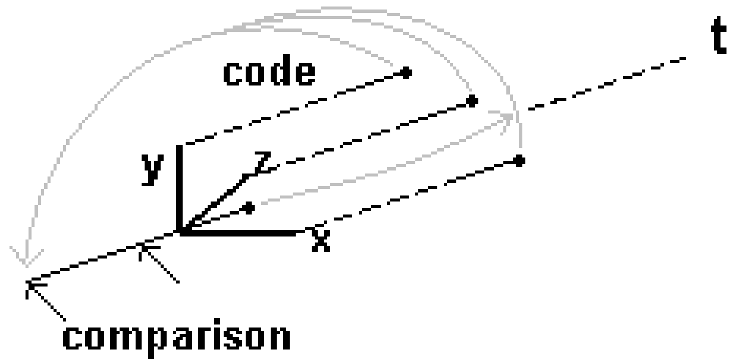
Template: If-Then-Else

These motions implement *if-then-else*.

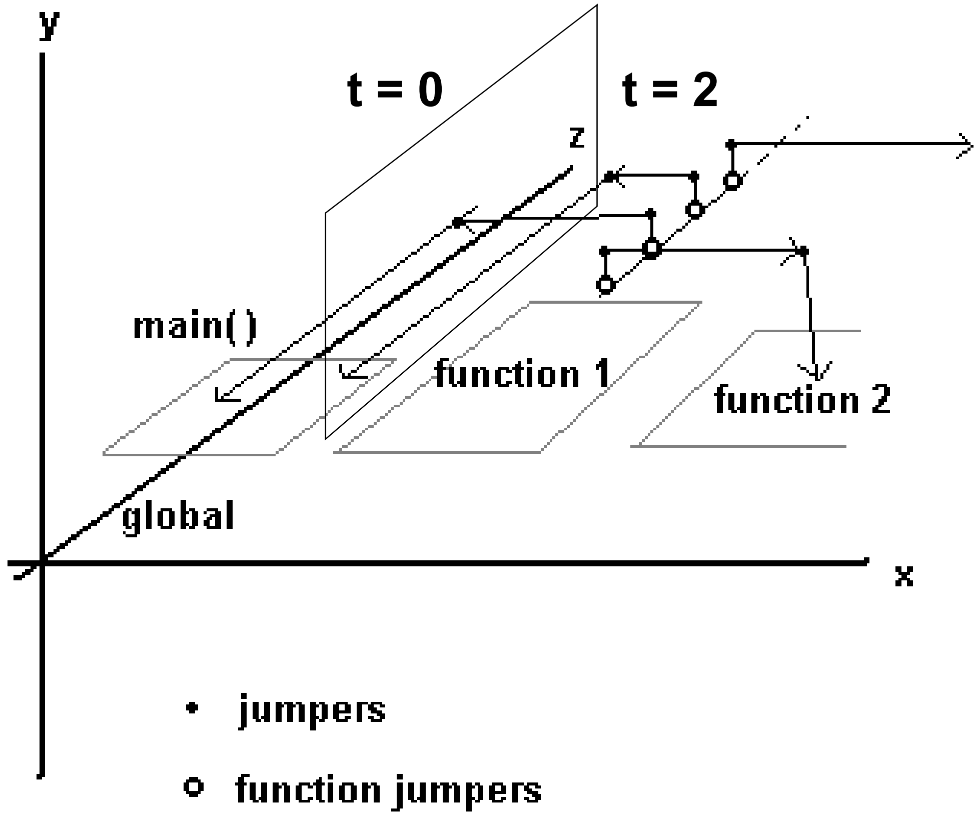


Template: While

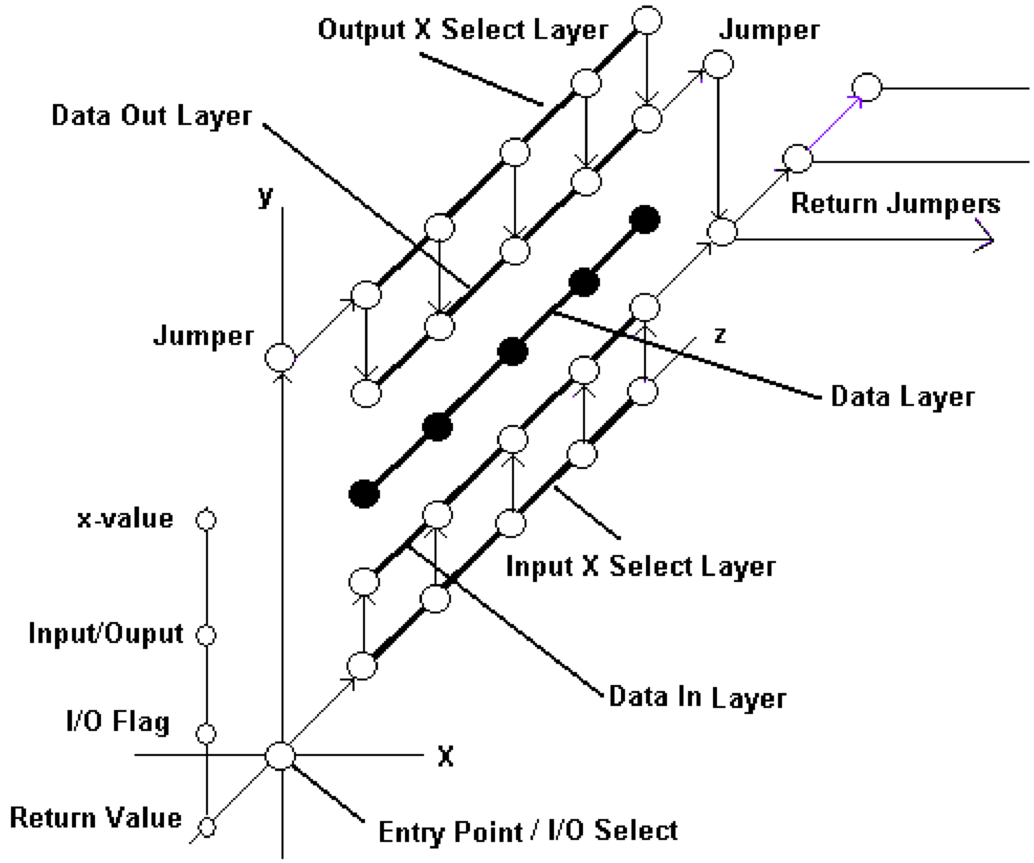
These motions implement *while.*



Template: Function



Template: Array



Compiling C into Perspexes

- Arrays (and any structures) are immune from buffer-overflow viruses because writing beyond the bounds of an array (or structure) requires that control jumps into empty space and empty space contains the default instruction *halt*.
- Variables and constants are related by a rotation! So the type system is a geometrical transformation.
- Variables and constants can be blended to produce intermediate objects. Can this be used to implement a novel form of constraint satisfaction?

Compiling C into Perspexes

- Programs can be averaged, blended, Fourier transformed, filtered, reconstructed, bent, twisted …
- Programs can undergo any physical transformation.
- In future I will examine genetic algorithms implemented in perspexes.

Summary

- $-\infty = -1/0$, $\infty = 1/0$, and $\Phi = 0/0$ are numbers.
- IEEE floating-point arithmetic treats $0/0$ incorrectly.
- IEEE floating-point arithmetic treats sign bits incorrectly.

I challenge you to accept the above three assertions or else to disprove them.

The force of this challenge is that you should accept the numbers $-\infty$, ∞, Φ.

Summary

- The Walnut Cake Theorem is more general (but weaker) than the Chinese Remainder Theorem.
- The Walnut Cake Theorem shows that many Turing computable things with sub-quadratic convergence, converge non-monotonically.
- This explains physical things such as puctuated equilibria in genetic evolution and the occurrence of paradigm shifts in the scientific literature.

Summary

- C code can be compiled into perspexes.
- Perspex programs can be subjected to any physical transformation.

Future Work

- Axiomatise transrational arithmetic.
- Examine genetic algorithms implemented in perspexes.