Transcomplex Topology and Elementary Functions

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Introduction

Introduction

The transnumber systems allow division by zero

Introduction

Earlier we used transreal calculus to extend Newtonian Physics so that it works at singularities

Introduction

In future we want to develop transcomplex calculus so that we can extend Quantum Electrodynamics and Relativistic Physics

Introduction

In preparation for that future work we now present transcomplex topology and transcomplex elementary functions

Transcomplex Numbers

Transcomplex Numbers

$$\mathbb{C}^T = \left\{ \frac{x}{y};\ x, y \in \mathbb{C} \right\}$$

Transcomplex Numbers

$$\mathbb{C}^T = \mathbb{C} \cup \left\{ \frac{x}{0};\ x \in \mathbb{C},\ |x| = 1 \right\} \cup \left\{ \frac{0}{0} \right\}$$

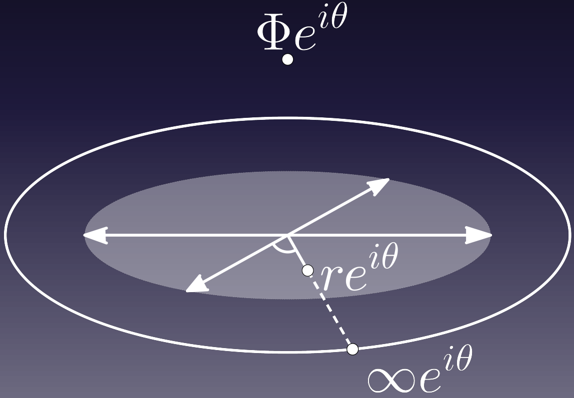
Transcomplex Numbers

$$\infty := \frac{1}{0} \qquad \Phi := \frac{0}{0}$$

Transcomplex Numbers

$$\mathbb{C}^T = \left\{ re^{i\theta}; r \in [0,\infty] \cup \{\Phi\}, \theta \in (-\pi, \pi] \right\}$$

Transcomplex Numbers



Transcomplex Topology

Transcomplex Topology

$$D := \{z \in \mathbb{C};\ |z| < 1\}$$

$$\overline{D} := \{z \in \mathbb{C};\ |z| \leq 1\}$$

Transcomplex Topology

$$\varphi : \mathbb{C}^T \setminus \{\Phi\} \rightarrow \overline{D} \subset \mathbb{C}^T$$

$$re^{i\theta} \mapsto \frac{1}{1+\frac{1}{r}} e^{i\theta}$$

Transcomplex Topology

$\mathbb{C}^T$ is a metric space

$$d : \mathbb{C}^T \times \mathbb{C}^T \to \mathbb{R}$$

$$d(z,w) = \begin{cases} 0, & \text{if } z = w = \Phi \\ 2, & \text{if } z = \Phi \text{ or else } w = \Phi \\ |\varphi(z) - \varphi(w)|, & \text{otherwise} \end{cases}$$

Transcomplex Topology

Transcomplex topology contains complex topology

Transcomplex Topology

$U \subset \mathbb{C}^T$ is open on $\mathbb{C}^T \implies U \cap \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$

$U \subset \mathbb{C}$ is open (in the usual sense) on $\mathbb{C} \implies U$ is open on $\mathbb{C}^T$

Transcomplex Topology

$\mathbb{C}^T$ is a separable space

$\mathbb{C}^T$ is a compact space

$\mathbb{C}^T$ is a complete space

$\mathbb{C}^T$ is a disconnected space

Φ is the unique isolated point of $\mathbb{C}^T$

Transcomplex Sequences

Transcomplex Sequences

Every sequence of transcomplex numbers has a convergent subsequence

Transcomplex Sequences

$\lim_{n\to\infty} x_n = L$ in $\mathbb{C}^T \iff \lim_{n\to\infty} x_n = L$ in the usual sense in $\mathbb{C}$

$\lim_{n\to\infty} x_n = \Phi \iff$ there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$

Transcomplex Limits

Transcomplex Limits

$\lim_{x \to x_0} f(x) = L$ in $\mathbb{C}^T \iff \lim_{x \to x_0} f(x) = L$ in the usual sense in $\mathbb{C}$

$\lim_{x \to x_0} f(x) = \Phi \iff$ there is a neighbourhood U of x_0 such that $f(x) = \Phi$ for all $x \in U \setminus \{x_0\}$

Transcomplex Continuity

Transcomplex Continuity

f is continuous in x_0 in $\mathbb{C}^T \Longleftrightarrow f$ is continuous in x_0 in the usual sense in $\mathbb{C}$

Elementary Functions

Elementary Functions

$$\begin{aligned} f : \mathbb{C}^T &\longrightarrow \mathbb{C}^T \\ z &\longmapsto a_n x^n + \cdots + a \end{aligned}$$

Elementary Functions

$$\begin{aligned} f : \mathbb{C}^T &\longrightarrow \mathbb{C}^T \\ re^{i\theta} &\longmapsto \exp\left(re^{i\theta}\right) \end{aligned}$$

where $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}$ if $\theta \in \{0, \pi\}$ and $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}(\cos(\infty\sin(\theta)) + i\sin(\infty\sin(\theta)))$ if $\theta \notin \{0, \pi\}$

Elementary Functions

$\exp(z) = e^z$ for every $z \in \mathbb{C}$

$\exp(-\infty) = 0$

$\exp(\infty) = \infty$

$\exp\left(\infty e^{i\theta}\right) = \Phi$ if $\theta \in (-\pi, \pi) \setminus \{0\}$

$\exp(\Phi) = \Phi$

Elementary Functions

$\exp$ is discontinuous in all infinities

Elementary Functions

The property

$$\exp(z + w) = \exp(z)\exp(w)$$

does not hold for all $z, w \in \mathbb{C}^T$

Elementary Functions

For example, let $z = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{0}$ and $w = \frac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i}{0}$. We have that $z+w = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{0} + \frac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i}{0} = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i}{0} = \frac{\sqrt{2}}{0} = \frac{1}{0} = \infty$ whence $\exp(z+w) = e^{\infty} = \infty$.

Elementary Functions

But $z = \infty e^{\frac{\pi}{4} i}$ and $w = \infty e^{-\frac{\pi}{4} i}$ whence $\exp(z) = \exp(\infty e^{\frac{\pi}{4} i}) = \Phi$, $\exp(w) = \exp(\infty e^{\frac{-\pi}{4} i}) = \Phi$. So $\exp(z)\exp(w) = \Phi \times \Phi = \Phi$. Therefore $\exp(z + w) \neq \exp(z)\exp(w)$.

Elementary Functions

$$\begin{aligned} f : \mathbb{C}^T &\longrightarrow \mathbb{C}^T \\ re^{i\theta} &\longmapsto \ln(r) + i\theta \end{aligned}$$

Elementary Functions

$\ln(z) = \infty$ for every transcomplex infinity z

Elementary Functions

The property $\ln(\exp(z)) = z$ does not hold for all $z \in \mathbb{C}^T$

If $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$ then $\ln(\exp(\infty e^{i\theta})) = \ln(\Phi) = \Phi \neq \infty e^{i\theta}$

$\ln(\exp(z)) = z$ holds in the other cases

Elementary Functions

$\exp(\ln(z)) = z$ does not hold for all $z \in \mathbb{C}^T$

If $\theta \in (-\pi, \pi] \backslash \{0\}$ then $\exp(\ln(\infty e^{i\theta}))$
$= \exp(\infty) = \infty \neq \infty e^{i\theta}$

$\exp(\ln(z)) = z$ holds in the other cases

Elementary Functions

For all $z, w \in \mathbb{C}^T$, $\ln(zw) = \ln(z) + \ln(w) + ki2\pi$ for some $k \in \mathbb{Z}$

In particular if the two conditions $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C} \setminus \{0\}$ do not hold simultaneously then $\ln(zw) = \ln(z) + \ln(w)$

Elementary Functions

$$z^w := \exp(w \ln(z)) \text{ for all } z, w \in \mathbb{C}^T$$

Elementary Functions

$$\begin{aligned} \sin : \mathbb{C}^T &\longrightarrow \mathbb{C}^T \\ z &\longmapsto \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} \end{aligned}$$

Elementary Functions

$$\begin{aligned} \cos : \mathbb{C}^T &\longrightarrow \mathbb{C}^T \\ z &\longmapsto \cos(z) = \frac{\exp(iz)+\exp(-iz)}{2} \end{aligned}$$

Elementary Functions

$$\sin^2(z) + \cos^2(z) = 1^z$$

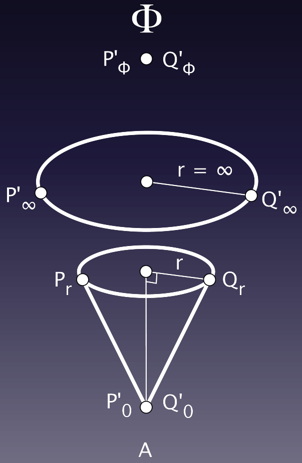
if and only if $z \in \mathbb{C}^T \setminus \{-i\infty, i\infty\}$

Elementary Functions

Transcomplex elementary functions contain complex and transreal elementary functions

Transcomplex Cone

Transcomplex Cone



Transcomplex Cone

- Angle is arc length divided by radius at all finite radii - including zero!
- Angle at unit radius is θ
- Angle wound at smaller radii is $k\pi + \theta$

Transcomplex Cone

- Angle at the apex is $0/0 = \Phi$
- Assuming continuity, angle at the apex is $0/0 = \Phi = \pm\infty\pi + \theta$
- This geometrical angle agrees with transreal powerseries and transreal trigonometry

Conclusion

Conclusion

- Transcomplex numbers can be expressed in exponential form
- Transcomplex elementary functions contain complex elementary functions

Conclusion

- Transcomplex topology is a metric space
- Transcomplex continuity and limits contain complex continuity and limits

Conclusion

- Transreal angles can be constructed geometrically
- Transreal angles contain real angles

Conclusion

- We have removed infinitely many division-by-zero errors from complex functions
- We now have the foundations to develop transcomplex calculus