Exact Numerical Computation of the Rational General Linear Transformations

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Aims

- To define transrational numbers.
- To parameterise the family of rational rotations.
- To specify exact computation of rotational sensor data.

Introduction

- **Computation:** General Linear.
- **Transrational:** Numbers.
  Arithmetic.
  Trigonometry.
- **Sensors:** Rotational.
- **Conclusion**
- **Questions**

Computation: General Linear

- General linear computations can be performed exactly using rational arithmetic - up to machine limits.
- However, no convenient parameterisation of the rational rotations currently exists, so we will develop one.
- We will use the parameterisation to specify exact computation of rotational sensor data.

Transrational: Numbers

- Nullity, $\Phi = 0/0$.
- Infinity, $\infty = 1/0$.
- Transrational numbers, $T = Q \cup \{\Phi, \infty\}$.

Transrational: Numbers

- The equivalence of transrational fractions n/d is given by the standard equivalence of rational arithmetic (EQ 1) and the standard sign convention of rational arithmetic (EQ 2).

$$n_1 d_2 = n_2 d_1 \quad \text{(EQ 1)}$$

$$\pm\mathrm{sgn}(n_1) = \mathrm{sgn}(n_2) \text{ and } \pm\mathrm{sgn}(d_1) = \mathrm{sgn}(d_2) \quad \text{(EQ 2)}$$

- These equations falsify all monstrous equivalences.

Transrational: Numbers

- Φ is disjoint from all other transrational numbers, so it is its own canonical form.
- ∞ is chosen as the canonical form of all fractions with a zero denominator, other than Φ.
- The standard canonical form of rational numbers is adopted.
- This completes the definition of the transrational numbers.

Transrational: Equivalence

- Here $x, y \in Z^{+}$.

$\equiv$	$\frac{x}{y}$	$\frac{-x}{y}$	$\frac{x}{-y}$	$\frac{-x}{-y}$	$\frac{0}{y}$	$\frac{0}{-y}$	$\frac{x}{0}$	$\frac{-x}{0}$	$\frac{0}{0}$
x/y	T	$F_{1,2}$	$F_{1,2}$	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$(-x)/y$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$x/(-y)$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$(-x)/(-y)$	T	$F_{1,2}$	$F_{1,2}$	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$0/y$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	F_2
$0/(-y)$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	F_2
$x/0$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	F_2
$(-x)/0$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	F_2
$0/0$	F_2	F_2	F_2	F_2	F_2	F_2	F_2	F_2	T

Transrational: Arithmetic

- The operations of rational arithmetic are applied syntactically to fractions with the results reduced to lowest terms as defined above.

$$(a, b) + (c, d) = (ad + bc, bd)$$

$$(a, b) \times (c, d) = (ac, bd)$$

$$-(a, b) = (-a, b)$$

$$(a, b)^{-1} = (b, a)$$

- Thus rational calculations produce the same result in transrational arithmetic and are disjoint from Φ and ∞.

Transrational: Arithmetic

- We assume $\infty > 0$.
- We adopt the ordering relationship of rational arithmetic applied syntactically and reduced to lowest terms. That is, with $x, y \in T$:

$$x > y \Leftrightarrow x - y > 0$$

- An ordering of T follows. Let $q = n/d$, $q \in Q$, then $\infty - q = 1/0 - n/d = (1d - 0n)/(0d) = d/0 = 1/0 = \infty > 0$.
- That is $\infty > q$. In other words, infinity is a point at infinity on the real number line.

Transrational: Arithmetic

- Nullity is not equal to any other transrational number, so it lies off the real number line augmented by infinity.

Transrational: Trigonometry

- Following Euclid, all of the integer roots of $p^2 + q^2 = r^2$ are given in terms of integers n, d by:

$$p = d^2 - n^2 \qquad q = 2dn \qquad r = d^2 + n^2$$

- We identify p and q respectively with the x and y Cartesian axes, then, without loss of generality, we let $d \geq 0$ and identify the triangle $p'q'r'$ with all triangles pqr identical over a positive dilatation:

$$p' = p/k \qquad q' = q/k \qquad r' = r/k$$

- Where k is the largest common factor of p, q, r; or else $k = 0$ when $r = 0$.

Transrational: Trigonometry

- We now see that the transrational numbers n/d encode all p', q', r'.
- Firstly, $\Phi = 0/0$ encodes the trivial solution $p' = q' = r' = 0$.
- Secondly, $\infty = 1/0$ encodes: $p' = -1$, $q' = 0$, $r' = 1$.

Transrational: Trigonometry

- Thirdly, rational n/d gives n and d relatively prime. Because kn, kd with $k \in Z$ gives:

$$
\begin{aligned}
p &= (kd)^2 - (kn)^2 = k^2(d^2 - n^2) \\
q &= 2(kd)(kn) = k^2(2dn) \\
r &= (kd)^2 + (kn)^2 = k^2(d^2 + n^2)
\end{aligned}
$$

- And k^2 is a common factor eliminated by the dilatation, so fractions with a common factor are of no interest.
- All fractions have now been considered, so the transrational numbers encode all p', q', r'.

Transrational: Numbers

- Transrational numbers are the solutions to the Euclidean equation.

Transrational: Trigonometry

- The paper defines a function *pqr* that generates p', q', r' from a transrational number t:

$$\mathrm{pqr}(t) \rightarrow (p', q', r')$$

- Analogously to the cyclicity of trigonometric functions over 2π, the transrational trigonometric functions are cyclic over 4.

- The transrational trigonometric functions are fixed for $t = \Phi$.

- The transrational trigonometric functions include the rational trigonometric functions.

Transrational: Trigonometry

- The inverse function *arcpqr* of *pqr* is:

$$\mathrm{arcpqr}(p', q', r') = \begin{cases} q'/(r' + p'), p \geq 0 \\ 2\mathrm{pty}(q') - q'/(r' - p'), p < 0 \end{cases}$$

- Here *pty* is the parity function of an integer z:

$$\mathrm{pty}(z) = \begin{cases} 1, z \geq 0 \\ -1, z < 0 \end{cases}$$

Transrational: Trigonometry

- The transrational trigonometric functions are:

$$\begin{aligned} \operatorname{cosq}(n/d) &= p'/r' & \operatorname{secq}(n/d) &= r'/p' \\ \operatorname{sinq}(n/d) &= q'/r' & \operatorname{cscq}(n/d) &= r'/q' \\ \operatorname{tanq}(n/d) &= q'/p' & \operatorname{cotq}(n/d) &= p'/q' \end{aligned}$$

Transrational: Trigonometry

- The inverse transrational trigonometric functions are:

$$\operatorname{arccosq}(p'/r') = \operatorname{arcsecq}(r'/p') = \operatorname{arcpqr}\left(p', \sqrt{(r'+p')(r'-p')}, r'\right)$$

$$\operatorname{arcsinq}(q'/r') = \operatorname{arccscq}(r'/q') = \operatorname{arcpqr}\left(\sqrt{(r'+q')(r'-q')}, q', r'\right)$$

$$\operatorname{arctanq}(q'/p') = \operatorname{arccotq}(p'/q') = \operatorname{arcpqr}\left(p', q', \sqrt{p'^2+q'^2}\right)$$

- These are exact when the integer square root is exact.

Transrational: Trigonometry

- When the above integer square root is not exact, the floor of the root gives a close rational approximation to the irrational solution.

Sensors: Rotational

- A real or virtual rotor in a sensor can be calibrated, down to atomic levels, in terms of the radius of the rotor, r', and the x- and y- displacements, p' and q', by using interferometry.

- The parameter t of the rotational position of the sensor is then given by:

$$t = \operatorname{arcpqr}(p', q', r') = \begin{cases} q'/(r'+p'), p \geq 0 \\ 2\operatorname{pty}(q') - q'/(r'-p'), p < 0 \end{cases}$$

Sensors: Rotational

- In a calibrated sensor the computation of t is exact, except for sensor error, so subsequent general linear computations are accurate precisely to the physical limits of the sensor.
- You can't have tighter error bounds than that!

Conclusion

- Defined transrational numbers.
- Parameterised the family of rational rotations.
- Specified exact computation of rotational sensor data.