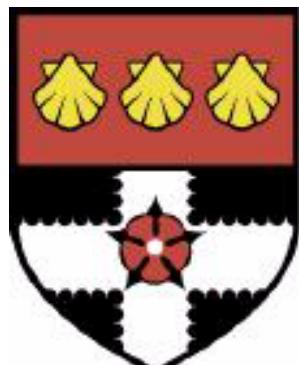


Exact Numerical Computation of the Rational General Linear Transformations

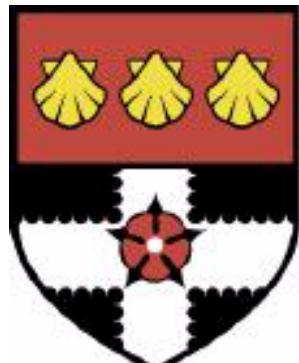
SPIE 2002

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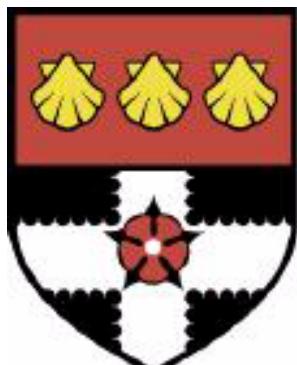
Aims

- To define transrational numbers.
- To parameterise the family of rational rotations.
- To specify exact computation of rotational sensor data.



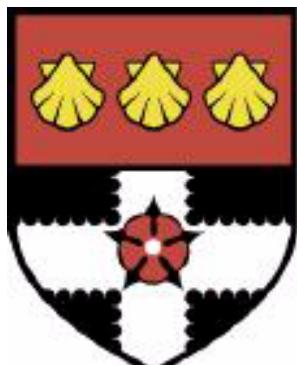
Introduction

- **Computation:** General Linear.
- **Transrational:** Numbers.
Arithmetic.
Trigonometry.
- **Sensors:** Rotational.
- **Conclusion**
- **Questions**



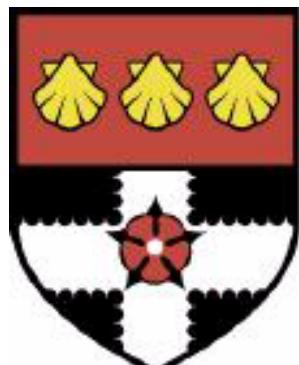
Computation: General Linear

- General linear computations can be performed exactly using rational arithmetic - up to machine limits.
- However, no convenient parameterisation of the rational rotations currently exists, so we will develop one.
- We will use the parameterisation to specify exact computation of rotational sensor data.



Transrational: Numbers

- Nullity, $\Phi = 0/0.$
- Infinity, $\infty = 1/0.$
- Transrational numbers, $T = Q \cup \{\Phi, \infty\}.$



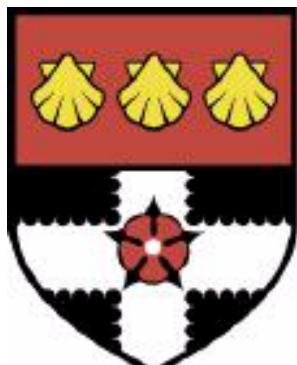
Transrational: Numbers

- The equivalence of transrational fractions n/d is given by the standard equivalence of rational arithmetic (EQ 1) and the standard sign convention of rational arithmetic (EQ 2).

$$n_1 d_2 = n_2 d_1 \quad (\text{EQ } 1)$$

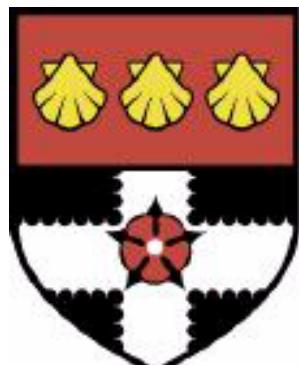
$$\pm \text{sgn}(n_1) = \text{sgn}(n_2) \text{ and } \pm \text{sgn}(d_1) = \text{sgn}(d_2) \quad (\text{EQ } 2)$$

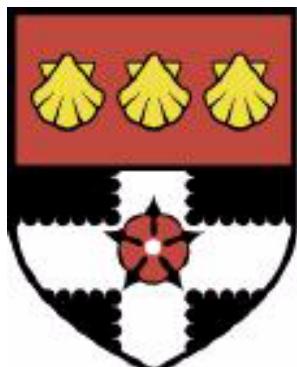
- These equations falsify all monstrous equivalences.



Transrational: Numbers

- Φ is disjoint from all other transrational numbers, so it is its own canonical form.
- ∞ is chosen as the canonical form of all fractions with a zero denominator, other than Φ .
- The standard canonical form of rational numbers is adopted.
- This completes the definition of the transrational numbers.





Transrational: Equivalence

- Here $x, y \in Z^+$.

\equiv	$\frac{x}{y}$	$\frac{-x}{y}$	$\frac{x}{-y}$	$\frac{-x}{-y}$	$\frac{0}{y}$	$\frac{0}{-y}$	$\frac{x}{0}$	$\frac{-x}{0}$	$\frac{0}{0}$
x/y	T	$F_{1,2}$	$F_{1,2}$	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$(-x)/y$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$x/(-y)$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$(-x)/(-y)$	T	$F_{1,2}$	$F_{1,2}$	T	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	F_2
$0/y$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	F_2
$0/(-y)$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	$F_{1,2}$	$F_{1,2}$	F_2
$x/0$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	F_2
$(-x)/0$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	$F_{1,2}$	T	T	F_2
$0/0$	F_2	F_2	F_2	F_2	F_2	F_2	F_2	F_2	T

Transrational: Arithmetic

- The operations of rational arithmetic are applied syntactically to fractions with the results reduced to lowest terms as defined above.

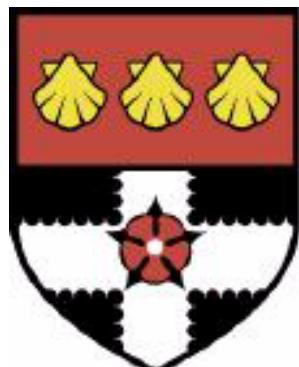
$$(a, b) + (c, d) = (ad + bc, bd)$$

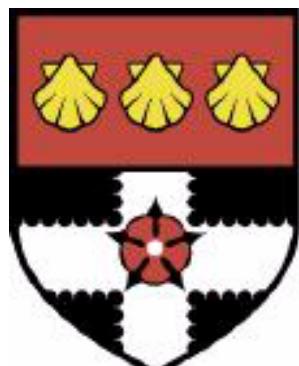
$$(a, b) \times (c, d) = (ac, bd)$$

$$-(a, b) = (-a, b)$$

$$(a, b)^{-1} = (b, a)$$

- Thus rational calculations produce the same result in transrational arithmetic and are disjoint from Φ and ∞ .





Transrational: Arithmetic

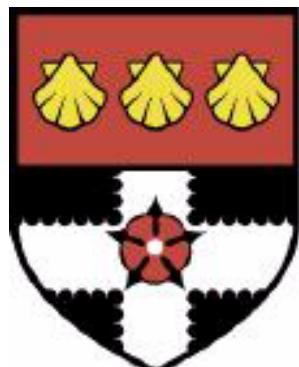
- We assume $\infty > 0$.
- We adopt the ordering relationship of rational arithmetic applied syntactically and reduced to lowest terms. That is, with $x, y \in T$:

$$x > y \Leftrightarrow x - y > 0$$

- An ordering of T follows. Let $q = n/d$, $q \in Q$, then $\infty - q = 1/0 - n/d = (1d - 0n)/(0d) = d/0 = 1/0 = \infty > 0$.
- That is $\infty > q$. In other words, infinity is a point at infinity on the real number line.

Transrational: Arithmetic

- Nullity is not equal to any other transrational number, so it lies off the real number line augmented by infinity.



Transrational: Trigonometry

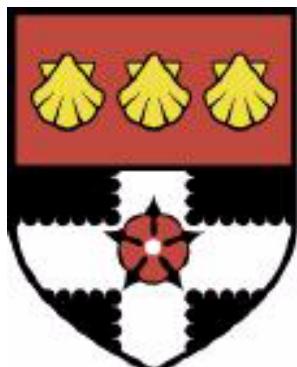
- Following Euclid, all of the integer roots of $p^2 + q^2 = r^2$ are given in terms of integers n, d by:

$$p = d^2 - n^2 \quad q = 2dn \quad r = d^2 + n^2$$

- We identify p and q respectively with the x and y Cartesian axes, then, without loss of generality, we let $d \geq 0$ and identify the triangle $p'q'r'$ with all triangles pqr identical over a positive dilatation:

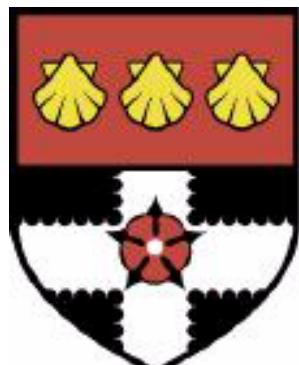
$$p' = p/k \quad q' = q/k \quad r' = r/k$$

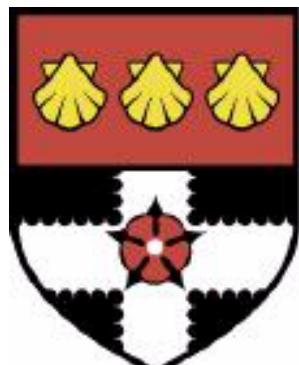
- Where k is the largest common factor of p, q, r ; or else $k = 0$ when $r = 0$.



Transrational: Trigonometry

- We now see that the transrational numbers n/d encode all p' , q' , r' .
- Firstly, $\Phi = 0/0$ encodes the trivial solution $p' = q' = r' = 0$.
- Secondly, $\infty = 1/0$ encodes:
 $p' = -1, q' = 0, r' = 1$.





Transrational: Trigonometry

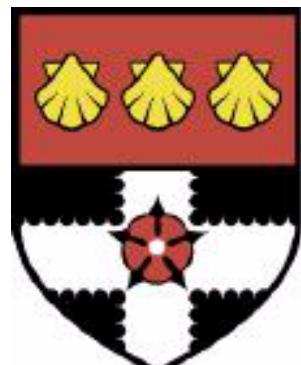
- Thirdly, rational n/d gives n and d relatively prime. Because kn, kd with $k \in \mathbb{Z}$ gives:

$$p = (kd)^2 - (kn)^2 = k^2(d^2 - n^2)$$

$$q = 2(kd)(kn) = k^2(2dn)$$

$$r = (kd)^2 + (kn)^2 = k^2(d^2 + n^2)$$

- And k^2 is a common factor eliminated by the dilatation, so fractions with a common factor are of no interest.
- All fractions have now been considered, so the transrational numbers encode all p', q', r' .



Transrational: Numbers

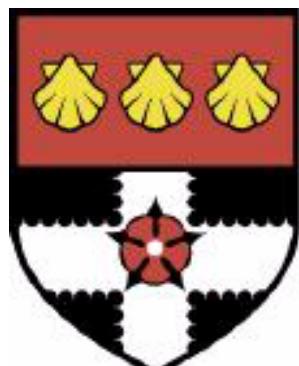
- Transrational numbers are the solutions to the Euclidean equation.

Transrational: Trigonometry

- The paper defines a function pqr that generates p' , q' , r' from a transrational number t :

$$pqr(t) \rightarrow (p', q', r')$$

- Analogously to the cyclicity of trigonometric functions over 2π , the transrational trigonometric functions are cyclic over 4.
- The transrational trigonometric functions are fixed for $t = \Phi$.
- The transrational trigonometric functions include the rational trigonometric functions.



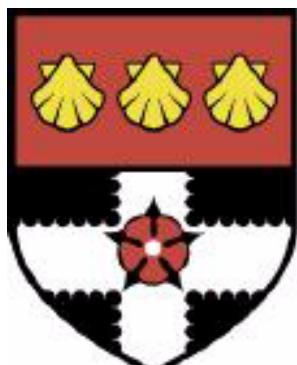
Transrational: Trigonometry

- The inverse function arcpqr of pqr is:

$$\text{arcpqr}(p', q', r') = \begin{cases} q'/(r' + p'), & p \geq 0 \\ 2\text{pty}(q') - q'/(r' - p'), & p < 0 \end{cases}$$

- Here pty is the parity function of an integer z :

$$\text{pty}(z) = \begin{cases} 1, & z \geq 0 \\ -1, & z < 0 \end{cases}$$



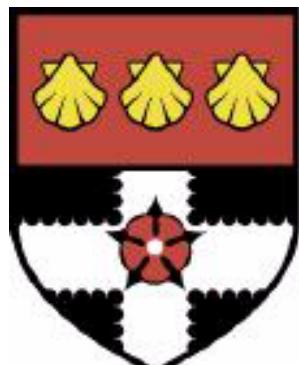
Transrational: Trigonometry

- The transrational trigonometric functions are:

$$\cosq(n/d) = p'/r' \quad \secq(n/d) = r'/p'$$

$$\sinq(n/d) = q'/r' \quad \cscq(n/d) = r'/q'$$

$$\tanq(n/d) = q'/p' \quad \cotq(n/d) = p'/q'$$



Transrational: Trigonometry

- The inverse transrational trigonometric functions are:

$$\arccosq(p'/r') = \operatorname{arcsecq}(r'/p') =$$

$$\operatorname{arcpqr}(p', \sqrt{(r' + p')(r' - p')}, r')$$

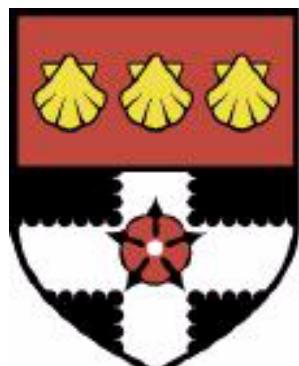
$$\arcsinq(q'/r') = \operatorname{arccscq}(r'/q') =$$

$$\operatorname{arcpqr}(\sqrt{(r' + q')(r' - q')}, q', r')$$

$$\arctanq(q'/p') = \operatorname{arccotq}(p'/q') =$$

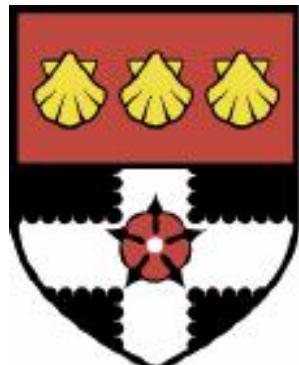
$$\operatorname{arcpqr}(p', q', \sqrt{p'^2 + q'^2})$$

- These are exact when the integer square root is exact.



Transrational: Trigonometry

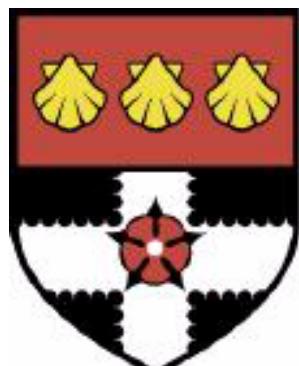
- When the above integer square root is not exact, the floor of the root gives a close rational approximation to the irrational solution.



Sensors: Rotational

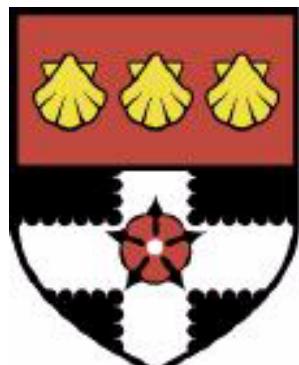
- A real or virtual rotor in a sensor can be calibrated, down to atomic levels, in terms of the radius of the rotor, r' , and the x - and y - displacements, p' and q' , by using interferometry.
- The parameter t of the rotational position of the sensor is then given by:

$$t = \text{arcPQR}(p', q', r') = \begin{cases} q'/(r' + p'), & p \geq 0 \\ 2pty(q') - q'/(r' - p'), & p < 0 \end{cases}$$



Sensors: Rotational

- In a calibrated sensor the computation of t is exact, except for sensor error, so subsequent general linear computations are accurate precisely to the physical limits of the sensor.
- You can't have tighter error bounds than that!



Conclusion

- Defined transrational numbers.
- Parameterised the family of rational rotations.
- Specified exact computation of rotational sensor data.

