Transreal Limits Expose Category Errors

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Agenda

- Advantages of transreal limits
- Transreal tangent
- Negative zero is a category error
- Transreal limits
- Value to science and society

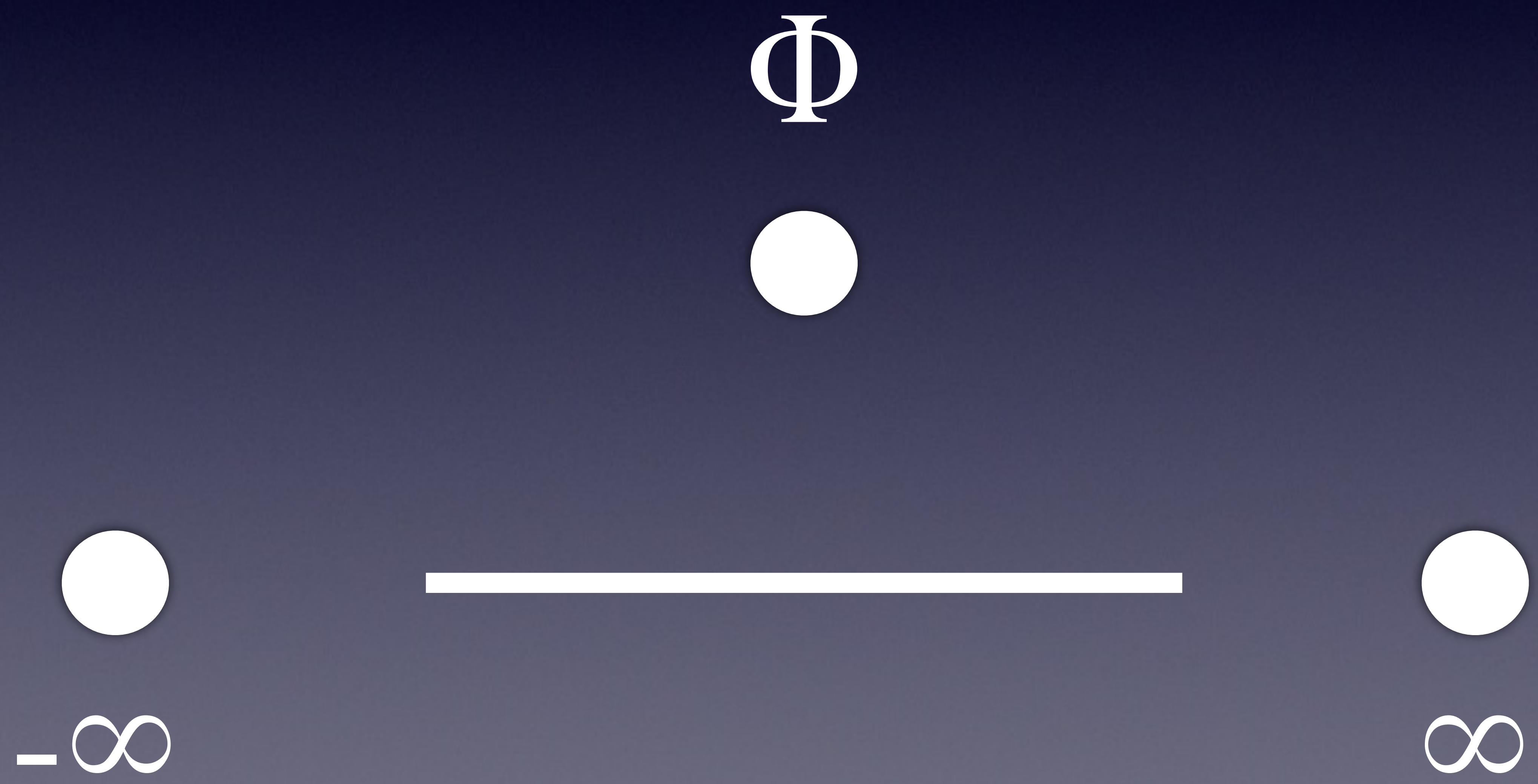
Advantages of Translimits

Translimits

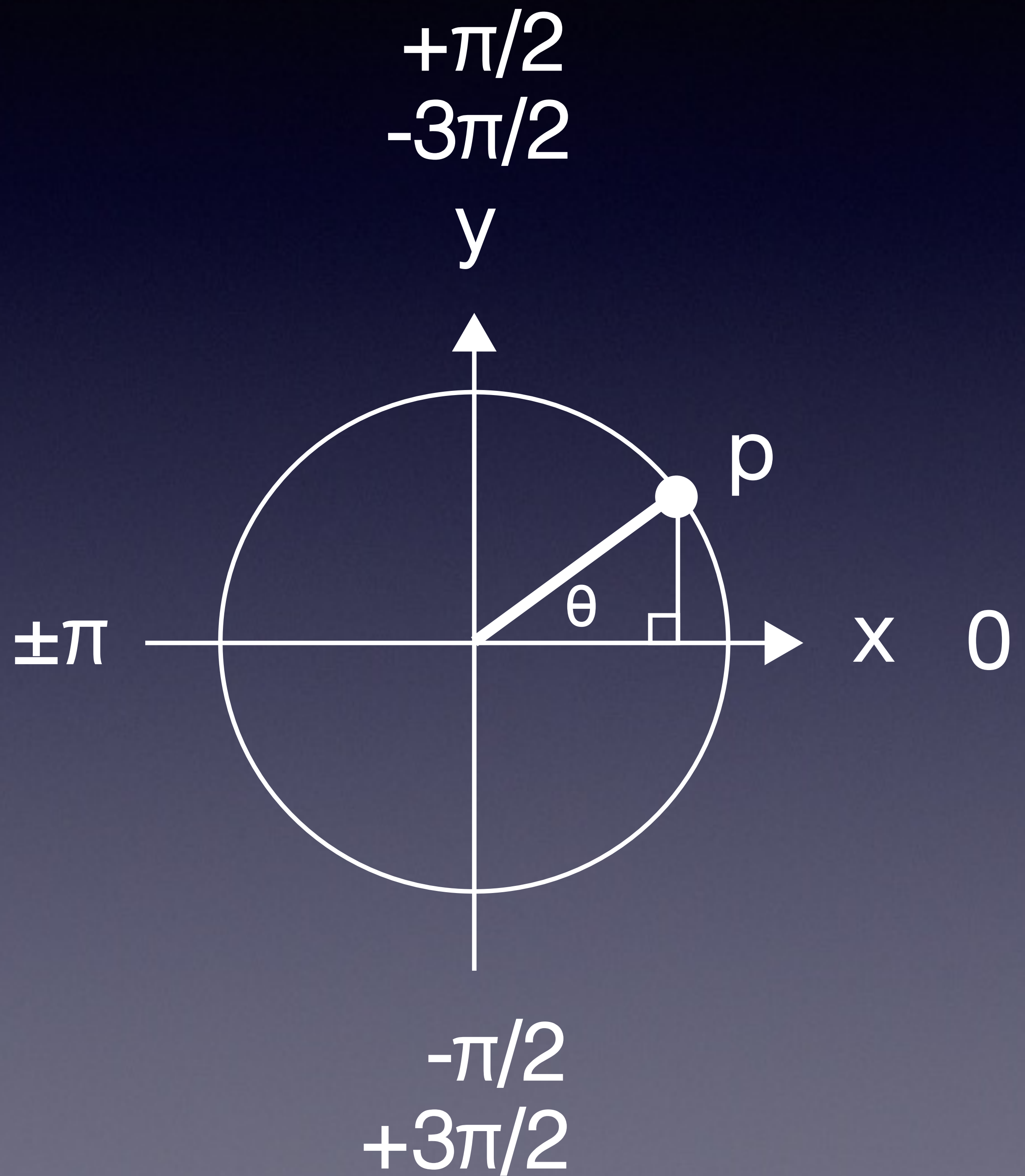
- Build on the foundation of transreal arithmetic
- Extend real analysis to transreal analysis
- Allow the solution of mathematical and physical problems at singularities
- Make mathematical software more reliable

Transtangent

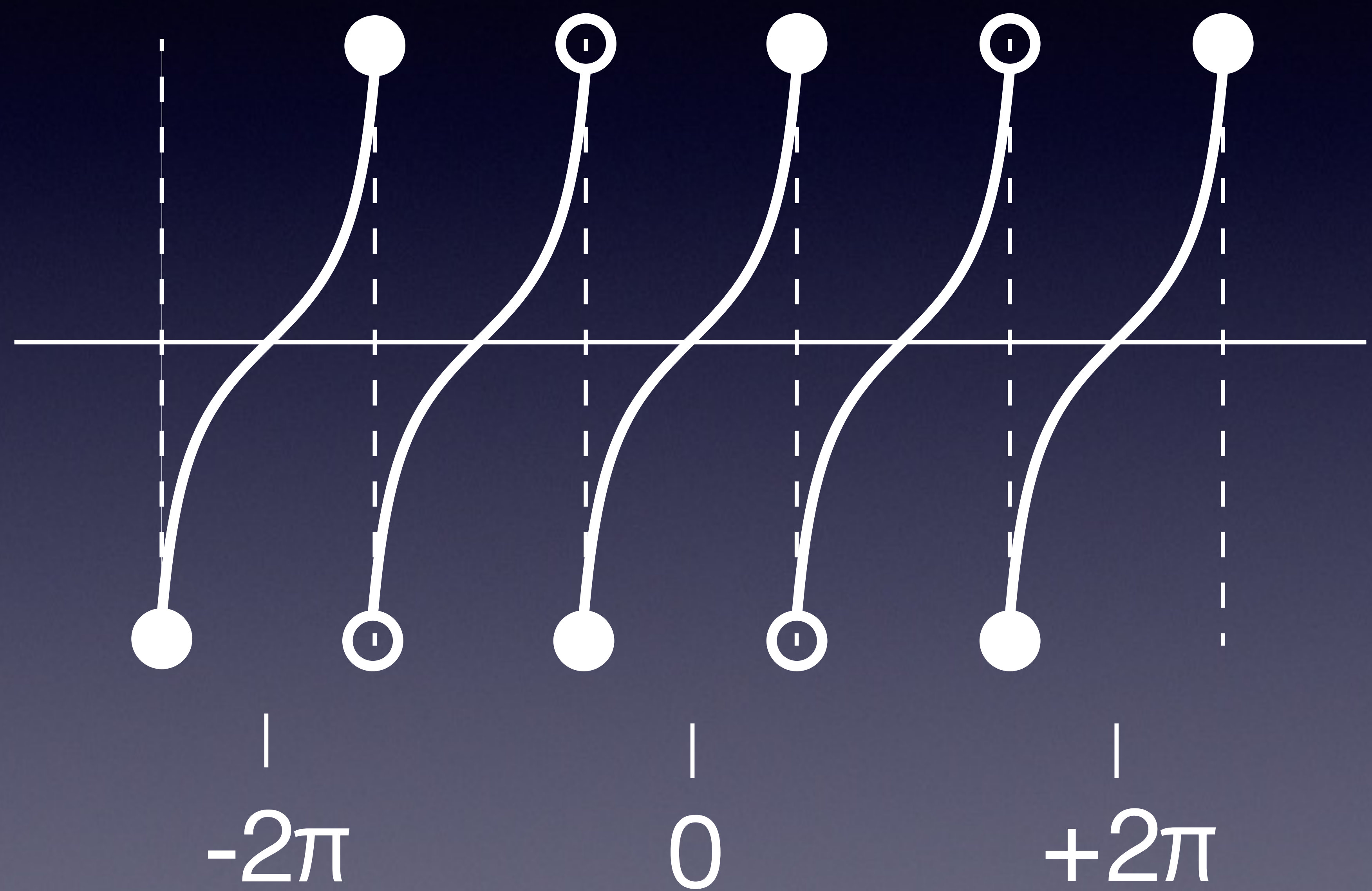
Transreal Number Line



Geometrical Construction



Finite Windings



Period 2π

Infinite Windings

No known geometrical construction of infinite windings so use power series evaluated with transreal arithmetic

$$\sin\infty = \infty - \frac{\infty^3}{3!} + \frac{\infty^5}{5!} - \dots$$

$$= \infty - \frac{\infty}{3!} + \frac{\infty}{5!} - \dots$$

$$= \infty - \infty + \infty - \dots$$

$$= \Phi + \infty - \dots$$

$$= \Phi$$

Infinite Windings

Similarly

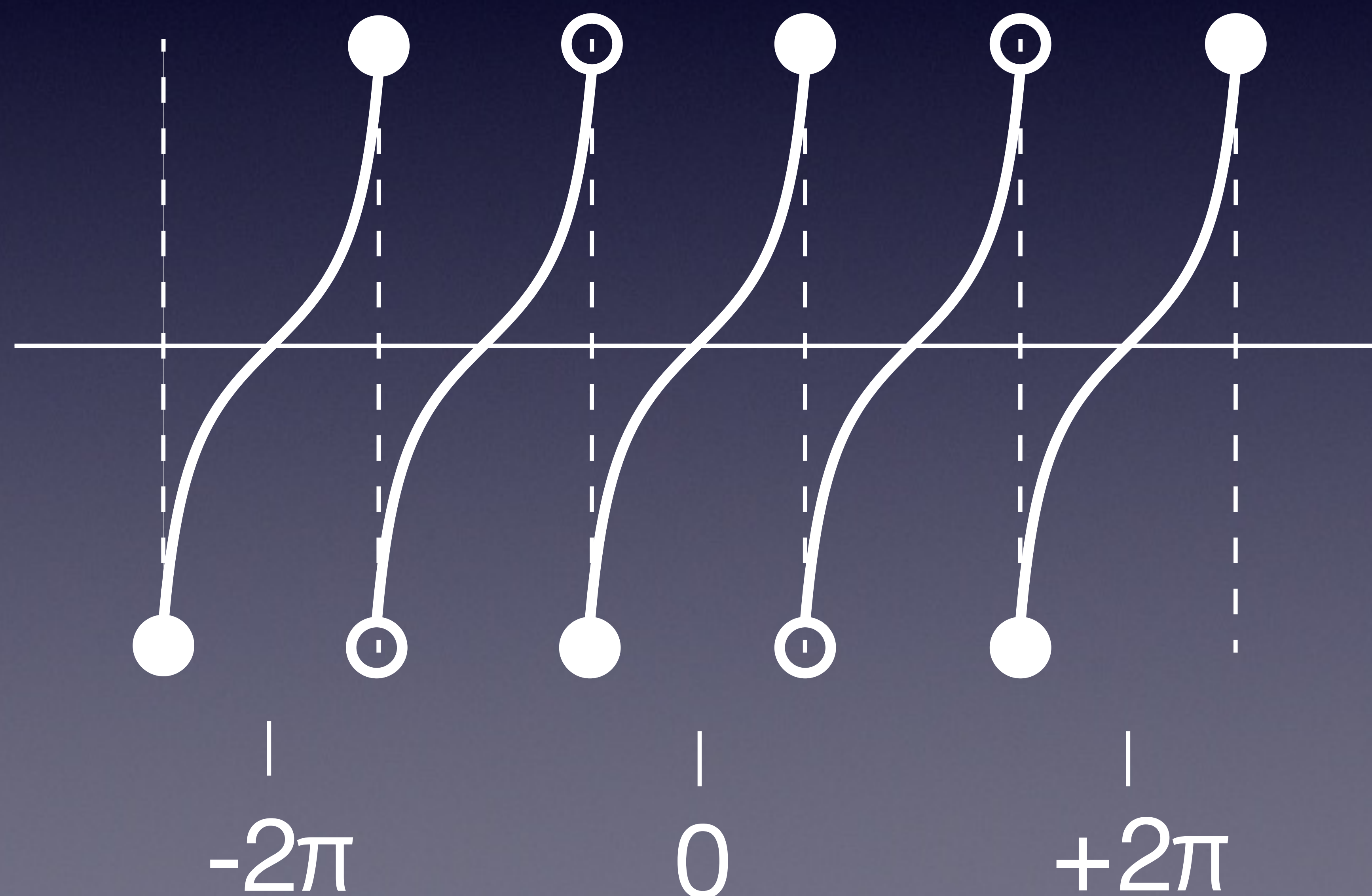
$$\sin\theta = \cos\theta = \tan\theta = \Phi$$

When

$$\theta \in \{-\infty, \infty, \Phi\}$$

Category Error

Dividing by minus zero instead of zero can be wrong!



Conjectures

- Definite, non-finite values of the tangent spread, by trigonometric identities, to many transreal and transcomplex, trigonometric functions
- Definite, non-finite, geometrical constructions spread to many transreal and transcomplex functions
- So transreal and transcomplex functions are less arbitrary than their ordinary counterparts

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$x = \Phi \Rightarrow \{\Phi\}$

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Φ is the unique isolated point of $\mathbb{R}^T$

Sequences

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$\lim_{n\to\infty} x_n = \Phi \Leftrightarrow$ there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$

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$\lim_{n \to \infty} x_n = L, \lim_{n \to \infty} z_n = L$ and $x_n \le y_n \le z_n \Rightarrow \lim_{n \to \infty} y_n = L$

Limit of Functions

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$\lim_{x\to x_0} f(x) = L \Leftrightarrow \lim_{n\to\infty} f(x_n) = L$ for all $(x_n)_{n\in\mathbb{N}}$

such that $x_n \neq x_0$ and $\lim_{n\to\infty} x_n = x_0$

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when $x, y \in \{-\infty, \infty\}$ and $x + y = \Phi$ do not occur simultaneously

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$$\lim_{n\to\infty}(x_n y_n) = xy$$

when $x, y \in \{0, \infty, -\infty\}$ and $xy = \Phi$

do not occur simultaneously

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Continuity of Functions

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f is continuous in $x_0 \Leftrightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$ for all $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x_0$

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f is continuous $\Leftrightarrow f^{-1}(B)$ is open, for all open B

Conclusion

- Real values of the transtangent are equal to real values of the tangent and have the same period of a half rotation
- Infinite values of the transtangent have a period of a whole rotation - the same as the period of the real values of both the real and transreal sine and cosine
- Negative zero is a category error
- Conjecture that transreal and transcomplex functions are less arbitrary than their ordinary counterparts

Conclusion

- The space of transreal numbers is a disconnected, separable, compact, Hausdorff space with nullity as the unique isolated point
- Translimits extend real analysis to transreal analysis
- Building up from transreal arithmetic to transreal limits is sound, going the other way, like IEEE floating-point arithmetic, is a category error

Value

Reach and Reliability

- Translimits allow the solution of mathematical and physical problems at singularities
- Make mathematical software more reliable

Translimits are a Foundation